HOW THE TALMUD DIVIDES AN ESTATE AMONG CREDITORS

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1. Introduction

A man dies leaving an estate that is too small to pay his debts. How much should each creditor get?

The Babylonian Talmud, a compendium of Jewish law that dates back 1800 years, gives the following example. Creditor 1 is owed 100, Creditor 2 is owed 200, and Creditor 3 is owed 300.

(1) If the estate is 100, each creditor gets 33 1/3.
(2) If the estate is 200, Creditor 1 gets 50, Creditors 2 and 3 get 75 each.
(3) If the estate is 300, Creditor 1 gets 50, Creditor 2 gets 100, Creditor 3 gets 150.

A literature stretching across 1500 years deals with the question: what algorithm is the Talmud using? Of course, as in any legal system, the answer must be based on the system's principles and precedents.

The problem was convincingly solved by two mathematicians at Hebrew University of Jerusalem, Robert Aumann and Michael Maschler, in the 1980's [1]. Later Marek Kaminski, a political scientist now at the University of California, Irvine, showed how, given the sizes of the debts, one can construct special-purpose glassware so that when an amount of liquid equal to the size of the estate is poured in, it will divide itself in the correct way [3].

The goal of this paper is to explain the Aumann-Maschler-Kaminski solution and its relation to game theory.

The estate division problem is related to bankruptcy, since the same issue arises in dividing the assets of a bankrupt person or corporation among creditors. As we shall see, the Aumann-Maschler-Kaminski solution is related to the Talmud's view of this and other situations in which money is owed.

My interest in the estate division problem grew out of a game theory course for undergraduates that I teach. When Aumann was awarded the Nobel Prize in Economics in 2005 for work in game theory, I read some of what was written about him. Most of his work is rather technical. Perhaps for that reason, journalists writing about Aumann tended to move quickly to the fact that he had solved an old problem from the Talmud, apparently because this was thought to be of general interest. I looked into what the problem was and found a story that opens in many directions.

Aumann learned of the Talmud's estate-division problem in 1980 or 1981 from his son Shlomo, who was studying at a Talmudic academy in Jerusalem and pointed his father to the relevant passage. Shlomo Aumann was killed in 1982 while serving in the Israeli army.

2. What is the Talmud?

The Talmud (more precisely, the Babylonian Talmud) consists of:

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- the Mishna (c. 200 CE), a written compendium of Judaism's Oral Law;
- the Gemara (c. 500 CE), a record of discussions by rabbis about the Mishna.

It is divided into 60 tractates, or books. The first printed version appeared in Italy around 1520, some 85 years after Gutenberg invented the printing press. A modern edition with English translation occupies 73 volumes.

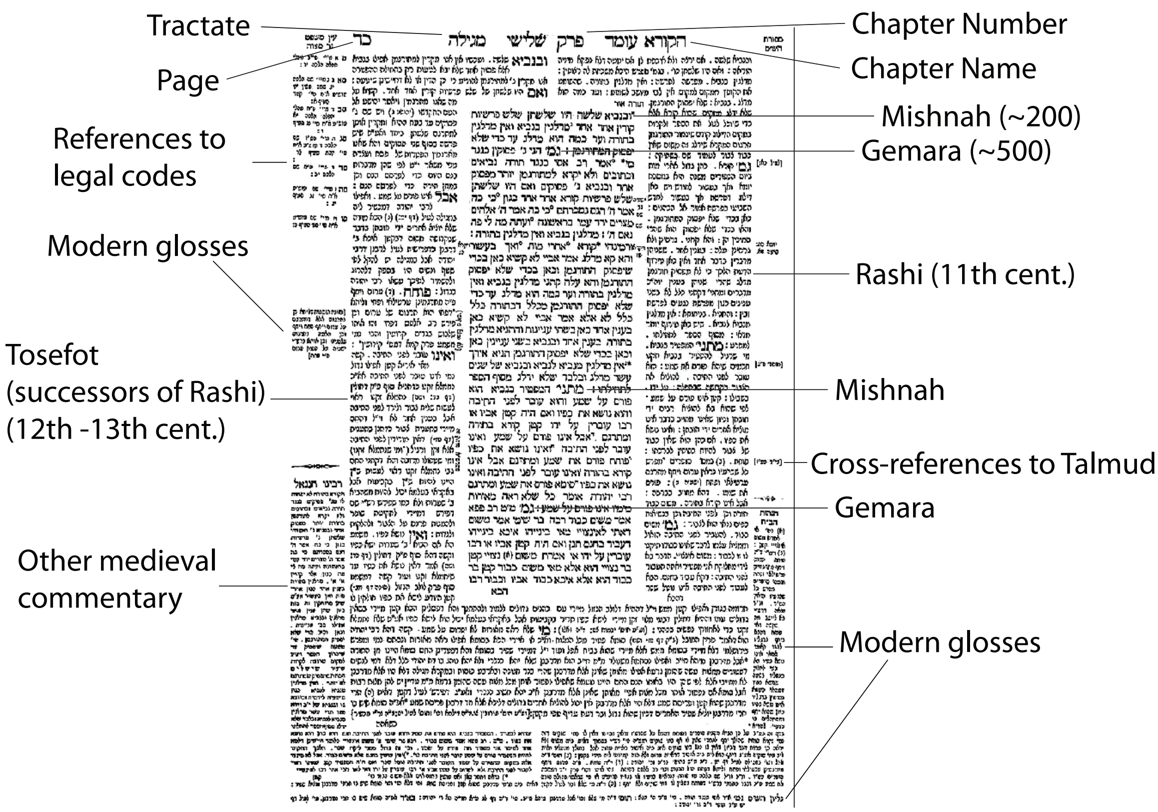

Figure 2.1.

Figure 2.1 shows a page of Talmud from a modern edition. The unusual form of the page dates back to the earliest printings. The header at the top of the page gives the tractate, in this case Megillah, or scroll, meaning the scroll of Esther; this tractate deals with laws concerning the reading of the Book of Esther at the holiday of Purim. The header also gives the chapter number and name, and the page number. Modern references to the Talmud give just the tractate and page number, followed by the letter a or b, meaning front or back of the page. The central block of text is portions of Mishna and the related Gemara, separated by colons. These texts are often fairly obscure; the Gemara is often in the form of notes on a discussion. The Mishna is in Hebrew, the Gemara in Aramaic. Commentary by Rashi (a French rabbi, 1040–1105) wraps around the central block at the upper right. The Talmud is considered to be largely incomprehensible without Rashi's commentary. Wrapping around the central block at the left is commentary by Rashi's successors in the 12th and 13th centuries. Outside these texts are more recent commentary, cross-references to related pages of Talmud, and references to codifications of Talmudic law.

The word "Mishna" is also used to refer to single portion of Mishna on a page of Talmud.

3. A problem from the Talmud

A man dies leaving

- an estate of size e;
- debts to Creditors $1, \ldots, n$ of $d_1, \ldots, d_n$;
- $e < d_1 + \cdots + d_n$.

How much should each creditor get?

A Mishna (Tractate Ketubot 93a) gives the answer described in the Introduction. ("Ketubot" is the plural of ketubah, which means marriage contract. This tractate includes laws about marriage and legal and financial aspects of family life.) Alfasi (a Moroccan rabbi, 1013–1103) wrote, "My predecessors discussed this Mishna and its Gemara at length, and were unable to make sense of it." Aumann and Maschler write in [1]: "Over two millennia, this Mishna has spawned a large literature. Many authorities disagree with it outright. Others attribute the figures to special circumstances, not made explicit in the Mishna. A few have attempted direct rationalizations of the figures as such, mostly with little success. One modern scholar, exasperated by his inability to make sense of the text, suggested errors in transcription. In brief, the passage is notoriously difficult."

An *estate division problem* is a pair $(e, (d_1, \ldots, d_n))$ with the following properties:

(1) $0 < d_1 \le d_2 \le \cdots \le d_n$.
(2) Let $d = d_1 + \ldots + d_n$. Then $0 < e < d$.

A *division* of the estate is an n-tuple $(x_1, \ldots, x_n)$ with $0 \le x_i$ for all i and $x_1 + \cdots + x_n = e$.

Here are some ideas about how an estate should be divided.

Proportional Division. Compute the fraction of the total debt that is owed each creditor, and assign her that fraction of the estate. (Following Aumann and Maschler, we use the pronoun "her" because in the Talmudic example, the creditors are women.) In other words, assign to creditor i the amount $\frac{d_i}{d}e$. Secular legal systems typically follow this idea, which treats each dollar of debt as equally worthy of payment. To most of us this approach seems obviously correct. Our Mishna appears to use this idea when $e = 300$.

Equal Division of Gains. Assign to each creditor the amount $\frac{e}{n}$. This method treats each *creditor* as equally worthy of payment. Our Mishna appears to use this idea when $e = 100$. Equal Division of Gains is not sensible if $d_1 < \frac{e}{n}$, since the first creditor (at least) will be paid more than she is owed. In other words, Equal Division of Gains is not sensible for large estates.

Constrained Equal Division of Gains. Give each creditor the same amount, but don't give any creditor more than her claim. In other words, choose a number a such that

$$\min(d_1, a) + \min(d_2, a) + \ldots \min(d_n, a) = e.$$

Then assign to creditor i the amount $\min(d_i, a)$. The number a exists and is unique because for fixed $(d_1, \ldots, d_n)$, the left-hand side is a function of a that maps the interval $[0, e]$ onto itself and is strictly increasing on this interval. This rule was adopted by Maimonides (1135–1204, born in Spain, worked in Morocco and Egypt) in his codification of Talmudic law, the Mishneh Torah, which is still considered canonical. Maimonides' choice is inconsistent with our Mishna (it produces equal division in all our cases).

Equal Division of Losses. Make each creditor take the same loss. The total loss to the creditors is $d - e$, so assign to creditor i the amount $d_i - (d - e)/n$. This is not sensible if $d_1 < (d - e)/n$, since Creditor 1's portion of the estate would be negative. In other words, Equal Division of Losses not sensible for small estates.

Constrained Equal Division of Losses. Make each creditor take the same loss, but don't make any creditor lose more than her claim.

The principle that losses should be shared equally was used by Maimonides in a different context. Suppose that at an auction, n bidders bid amounts $b_1 < b_2 < \cdots < b_{n-1} < b_n$. The object is sold to the highest bidder for the price b_n. If for some reason the highest bidder reneges, the object is sold to the second-highest bidder for b_{n-1}. The highest bidder's reneging has cost the seller the difference between the two bids, $b_n - b_{n-1}$. Maimonides says that the highest bidder is obligated to pay the seller this amount. Now suppose that all n bidders renege. This costs the seller b_n, the amount he should have sold the object for. Maimonides says that each bidder must pay the amount $\frac{b_n}{n}$ to the seller. The bidders lose equal amounts to cover what the seller should have gained.

Actually, Maimonides just gives a numerical example. In his example, equal payments by each bidder would result from either the Equal Division of Losses principle or the Constrained Equal Division of Losses principle. We can guess, based on Maimonides' adoption of Constrained Equal Division of Gains for division of estates, that what he had in mind was Constrained Equal Division of Losses.

4. The Aumann-Maschler Solution

Aumann and Maschler's solution to the Talmud's estate-division problem was based on another Mishna and an issue dealt with in Gemara.

4.1. **The Contested Garment Rule.** The relevant Mishna is from Tractate Bava Metzia 2a: "Two hold a garment; one claims it all, the other claims half. Then the one is awarded three-fourths, the other one-fourth."

("Bava Metzia" means middle gate. It deals with civil law, including property law. The name refers to the gates of a city, where markets were located.)

Rashi explains the reasoning. The one who claims half concedes that half belongs to the other. Therefore only half is in dispute. It is split equally between the two claimants.

Alfasi, in his commentary mentioned earlier, says that Rabbi Hai Gaon suggested without giving details that the Mishna about estate division should be explained using Bava Mezia 2a. Hai Gaon (939–1083) worked in what is today the Iraqi city of Falujah.

The second relevant passage is from Gemara in Tractate Yevamot 38a. ("Yevamot" is the plural of yibum, levirate marriage, i.e., the requirement that a widow marry her deceased husband's brother. This requirement is found in Deuteronomy 25:5-6.) It deals with a rather complicated family tree, illustrated in Figure 4.1.

Mr. B dies childless. His widow, as is required, marries his brother, C. C already has two sons, c_1 and c_2, by his first wife. Eight months later B's widow gives birth to a son, b, whose father is therefore doubtful. Next C dies. Finally, A, the father of B and C dies. How is A's estate to be divided among his grandchildren b, c_1, and c_2?

Young Mr. b says: Half of A's estate goes to A's son B and half to A's son C. I am B's only son, so I get his half. C's half should be divided between c_1 and c_2.

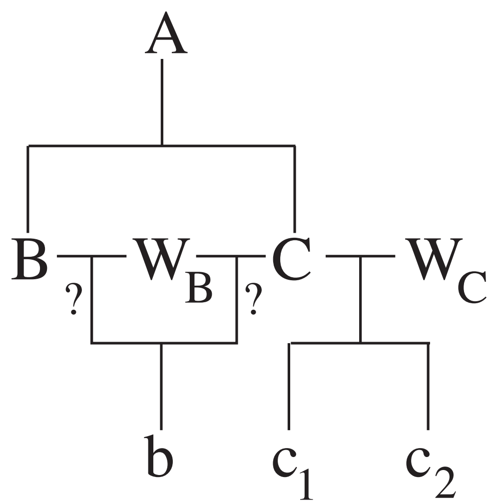

FIGURE 4.1. W_B is B's widow. W_C is C's first wife.

C's sons c_1 and c_2 say: B had no children, and C had three sons. Therefore the entire estate goes to C, and then is divided equally among the three grandchildren.

The Talmud's decision: c_1 and c_2 are treated as one claimant, b as another. The half of the estate that b concedes is not his goes to c_1 and c_2. The third of the estate that c_1 and c_2 concede is not theirs goes to b. The remainder of the estate, 1/6, is split equally: 1/12 to c_1 and c_2, 1/12 to b. Thus b gets 5/12 of the estate, and c_1 and c_2 get 7/12 to split.

Neither passage of Talmud treats a situation exactly analogous to an estate with creditors: there all claims are valid, whereas in these two passages, both claims cannot be valid. Nevertheless, applied to an estate with two creditors, we get:

Contested Garment Rule. Consider an estate division problem with two creditors: $0 < d_1 \leq d_2$, $0 < e < d_1+d_2$. Creditor 2 concedes $\max(e-d_2, 0)$ to Creditor 1. Creditor 1 concedes $\max(e-d_1, 0)$ to Creditor 2. The remainder of the estate, $e - \max(e-d_1, 0) - \max(e-d_2, 0)$, is divided equally. Thus Creditor 1 receives

$$\max(e - d_2, 0) + \frac{1}{2}(e - \max(e - d_1, 0) - \max(e - d_2, 0)).$$

Creditor 2 receives

$$\max(e - d_1, 0) + \frac{1}{2}(e - \max(e - d_1, 0) - \max(e - d_2, 0)).$$

4.2. **Aumann and Maschler's Theorem.** Here's our Mishna again. There are three debts, $d_1 = 100$, $d_2 = 200$, $d_3 = 300$.

(1) If $e = 100$, each creditor gets 33 1/3.
(2) If $e = 200$, Creditor 1 gets 50, Creditors 2 and 3 get 75 each.
(3) If $e = 300$, Creditor 1 gets 50, Creditor 2 gets 100, Creditor 3 gets 150.

Aumann and Maschler observed that each of these divisions is consistent with the Contested Garment Rule in the following sense. If any two creditors use the Contested Garment Rule to split the amount they were jointly awarded, each will get the amount she was actually awarded.

For example, look at the division with $e = 200$. Creditors 1 and 2 between them are awarded 125. Consider an estate of size 125 with two claims on it, $d_1 = 100$ and $d_2 = 200$ (these were the original claims of Creditors 1 and 2). According to the Contested Garment Rule, Creditor 1 concedes 25 to Creditor 2, and Creditor 2 concedes nothing to Creditor 1. The remaining 100 is split equally between the two. Thus Creditor 1 gets 50 and Creditor 2 gets 75. These are the amounts that the Mishna awarded them.

In an estate division problem $(e, (d_1, \dots, d_n))$, a division $(x_1, \dots, x_n)$ of the estate is *consistent with the Contested Garment Rule* if, for each pair (i, j), (x_i, x_j) is exactly the division produced by the Contested Garment Rule applied to an estate of size $x_i + x_j$ with debts d_i and d_j.

Aumann and Maschler proved:

Theorem 4.1 (Aumann-Maschler)**.** *In any estate division problem, there is exactly one division of the estate that is consistent with the Contested Garment Rule.*

This is the division that the Talmud presumably has in mind. Aumann and Maschler's proof was algebraic. We won't give it, since a nicer one appeared some years later.

4.3. **Another look at the Contested Garment Rule.** Let's look more closely at the Contested Garment Rule for an estate with two creditors.

(1) If $e \le d_1$, neither creditor concedes anything to the other, so the estate is split equally: each creditor gets $\frac{e}{2}$. Each additional dollar of estate value produces an equal gain for each creditor. See Figure 4.2.

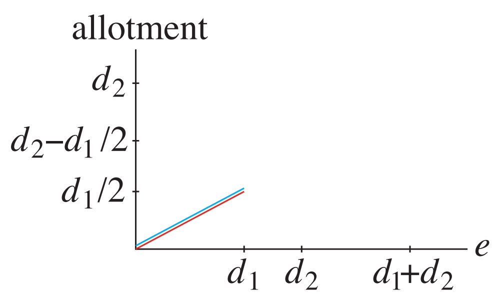

FIGURE 4.2. Division of the estate when $e \le d_1$.

(2) If $d_1 < e \le d_2$, $e - d_1$ is conceded by Creditor 1 to Creditor 2, nothing is conceded by Creditor 2 to Creditor 1, and the remainder, d_1, is split equally.

$$\text{Creditor 1: } \frac{d_1}{2}. \qquad \text{Creditor 2: } (e - d_1) + \frac{d_1}{2}.$$

See Figure 4.3. When $e = d_1$, the estate is split equally, so each Creditor has a gain of $\frac{d_1}{2}$. Thereafter each additional dollar of estate value goes to Creditor 2. When e reaches d_2, Creditor 1 gets $\frac{d_1}{2}$ and Creditor 2 gets $d_2 - \frac{d_1}{2}$, so each creditor has a loss of $\frac{d_1}{2}$ relative to the debt she is owed. Previously Creditor 2's loss was larger.

(3) If $d_2 < e < d_1 + d_2$, $e - d_1$ is conceded by Creditor 1 to Creditor 2, $e - d_2$ is conceded by Creditor 2 to Creditor 1, and the remainder, $e - (e - d_1) - (e - d_2) = d_1 + d_2 - e$,

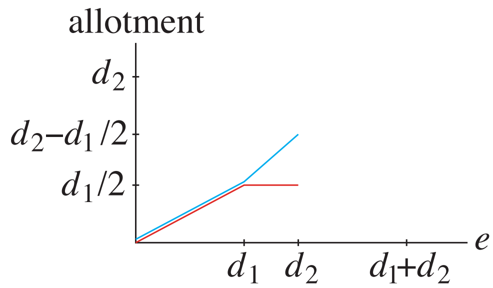

FIGURE 4.3. Division of the estate when $e \leq d_2$.

is split equally.

$$\text{Creditor 1: } e - d_2 + \frac{1}{2}(d_1 + d_2 - e) = \frac{d_1}{2} + \frac{1}{2}(e - d_2).$$

$$\text{Creditor 2: } e - d_1 + \frac{1}{2}(d_1 + d_2 - e) = d_2 - \frac{d_1}{2} + \frac{1}{2}(e - d_2).$$

We saw previously that when $e = d_2$, each creditor has a loss of $\frac{d_1}{2}$ relative to the debt she is owed. The part of the estate above d_2 is split equally, so the two creditors' losses remain equal.

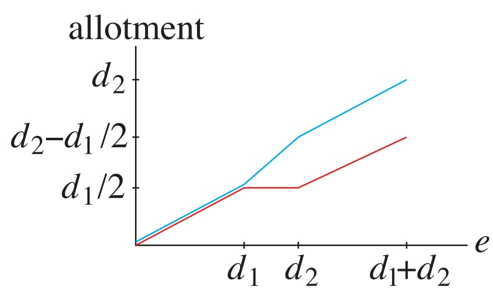

FIGURE 4.4. Division of the estate for all $e \leq d_1 + d_2$.

We conclude that *the Contested Garment Rule linearly interpolates between Equal Division of Gains for $e \leq d_1$ (small estates) and Equal Division of Losses for $d_2 \leq e$ (large estates).*

4.4. **More than half is like the whole.** Aumann and Maschler suggest that the Contested Garment Rule is perhaps related to the Talmudic principle that "more than half is like the whole, whereas less than half is like nothing." This principle says that the dividing line between two approaches to a problem is at the number one-half.

For example, in Talmudic law, a lender normally has an automatic lien on a borrower's property. However, in some cases, if the property is worth less than half the loan and the borrower is unable to repay it, the lender may not take the borrower's property (Arakhin 23b). Rashi explains that since the property is grossly inadequate to repay the loan, the lender has presumably relied not on the property but on the borrower's character for repayment, so the lender has no lien on the borrower's property.

In other words, if the property is worth less than half the loan and the borrower, despite his character, defaults, then the lender does not expect the loan to be repaid, so any repayment the lender receives is a gain relative to her expectation. If the property is worth more than half the loan, the lender expects the loan to be repaid, so any repayment she does not receive is a loss relative to her expectation.

The Contested Garment Rule can be seen as a sophisticated alternative to "more than half is like the whole." One principle is used to divide a small estate, and another is used to divide a large estate, but in between, one linearly interpolates between the two approaches.

(Arakhin 23b does not mean that the borrower is free of the obligation to repay the loan. Talmudic law does not have a concept of bankruptcy. If someone is unable to repay a loan, the lender cannot be forced to cancel it; it is the community's responsibility to help its destitute members, not the lender's. Should the borrower's circumstances improve, he must repay the lender.)

5. Kaminski's Proof of the Aumann-Maschler Theorem using glassware

Figure 5.1 is a schematic diagram of a piece of glassware. There are two glasses, of volumes

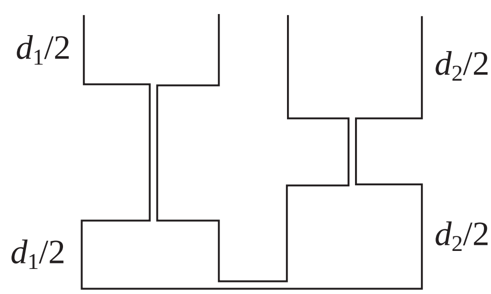

Figure 5.1.

d_1 and d_2. Each glass is equally divided between top and a bottom, connected by a small stem. In addition, a glass tube connects the bottoms of the two glasses. We will assume the stems and tube are very narrow, so that their volumes are negligible. The first glass represents the claim of Creditor 1, the second the claim of Creditor 2.

Suppose one pours a volume e of liquid, $0 < e < d_1 + d_2$, into this glassware. The liquid represents the estate. Because of the connecting tube at the bottom, the liquid will rise to the same height in both glasses. If you ignore the liquid in the stems and the connecting tube, you will see that each creditor gets the amount to which she is entitled by the Contested Garment Rule. For $e \leq d_1$, the liquid divides equally. For $d_1 < e \leq d_2$, Creditor 1 gets $\frac{d_1}{2}$, and the rest goes to Creditor 2. For $d_2 < e < d_1 + d_2$, each glass fills to within the same distance of the top, so each creditor has an equal loss.

The Aumann-Maschler Theorem can now be proved using more elaborate glassware. Given claims $d_1, \ldots, d_n$, construct the glassware shown in Figure 5.3. Pour in an amount e of liquid. It will rise to the same height in each glass. Since the height is the same in each pair of glasses, this division is consistent with the Contested Garment Rule. The division is unique: if we raise the height in one glass, in order to stay consistent with the Contested Garment Rule we must raise the height in all, so the total amount of liquid will increase.

Kaminski learned about the Talmud's estate division problem in a class taught by the game theorist Peyton Young at the University of Maryland. Young was explaining his concept of "parametric representation" of allocation methods. Kaminski writes [4]: "Sitting in class, I was repeatedly failing to visualize the parametric representation of the Talmudic solution, and, displeased with myself, I stopped listening and started thinking about an alternative.

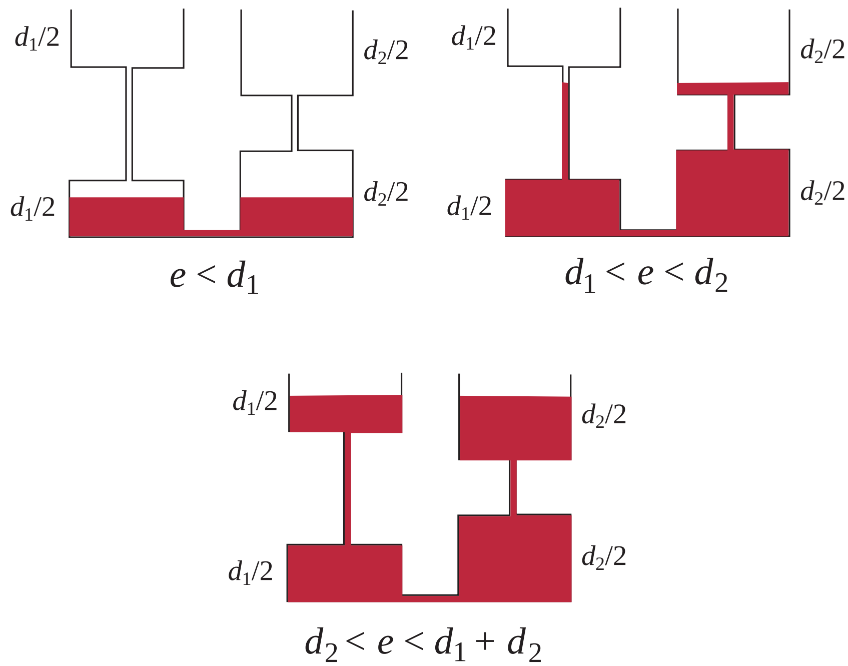

FIGURE 5.2.

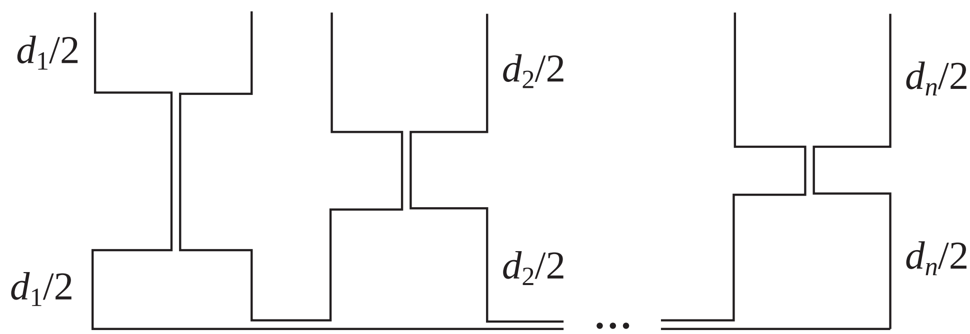

FIGURE 5.3.

The 'hydraulic' idea came to my mind in one of those unexplainable flashes. Later, I proved that in fact it is closely related to parametric representation."

6. THE AUMANN-MASCHLER THEOREM AND GAME THEORY

A *cooperative game* consists of

(1) a set of players $\{1, \ldots, n\}$;
(2) a value $V > 0$ to be divided among the players;
(3) a value function v from the power set of $\{1, \ldots, n\}$ into the nonnegative real numbers.

In this context, a subset S of $\{1, \ldots, n\}$ is called a *coalition*. The number $v(S)$ is interpreted as the part of the value V that the coalition S can get for itself no matter what the other players do. Because of this interpretation, the value function v is required to satisfy the following conditions.

(1) $v(\emptyset) = 0$.

(2) $v(\{1,\ldots,n\}) = V$.
(3) If S_1 and S_2 are disjoint, then $v(S_1) + v(S_2) \le v(S_1 \cup S_2)$.

An *allocation* of the value V to the players is a vector $x = (x_1,\ldots,x_n)$ such that all $x_i \ge 0$ and $x_1 + \cdots + x_n = V$. The problem of cooperative game theory is to choose the allocation. There are various ideas about how to do it. We will only discuss one of them..

Given an allocation x, the coalition S achieves the *excess* $e(x,S) = \sum_{j\in S} x_j - v(S)$. Coalitions with low excess will presumably complain that they have been treated unfairly and won't agree to the allocation. Perhaps one should choose x to avoid small excesses as much as possible and thus minimize the complaining.

More precisely, given an allocation x, calculate all $2^n - 2$ excesses $e(x,S)$. (We ignore the empty set and the set $\{1,\ldots,n\}$.) Order them from smallest to largest to form an *excess vector* $e \in \mathbb{R}^{2^n-2}$.

Given two excess vectors we can ask which precedes the other in the lexicographic ordering, which is defined as follows. Let $x = (x_1,\ldots,x_{2n-2})$ and $y = (y_1,\ldots,y_{2n-2})$ be two excess vectors, and suppose they first differ in the ith place. Then x precedes y in the lexicographic ordering if $x_i < y_i$.

- Example: $(1,2,4,5)$ precedes $(2,1,2,7)$.
- Example: $(2,2,2,7)$ precedes $(2,2,3,6)$.

Definition 6.1. The *nucleolus* of a cooperative game is the allocation whose excess vector is last in the lexicographic ordering.

Theorem 6.2. *Every cooperative game has a unique nucleolus.*

To find the nucleolus of a cooperative game, start with any allocation and adjust it to make one whose excess vector follows the excess vector of the first in the lexicographic ordering. When you can't go farther, you have found the nucleolus.

What does this have to do with estate division problems? Associated with any estate division problem is a cooperative game. The value to be divided is the estate e. To define the value function, assume that any coalition can guarantee itself the larger of 0 and the amount that remains if all other creditors are paid in full.

Let's work this out for the second example in our Mishna. A man dies leaving an estate of 200. There are three creditors with claims of 100, 200, and 300. Any coalition can guarantee itself the larger of 0 and whatever is left after those not in the coalition are paid in full. Therefore

$$v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0,\ v(\{1,2\}) = 0,\ v(\{1,3\}) = 0,\ v(\{2,3\}) = 100.$$

We claim that the nucleolus of this game is the allocation proposed by the Mishna: $(50,75,75)$. To see this we consider the following table of excesses for an arbitrary allocation (x_1,x_2,x_3) and for the allocation proposed by the Mishna.

S	$v(S)$	$e((x_1,x_2,x_3),S)$	$e((50,75,75),S)$
{1}	0	x_1	50
{2}	0	x_2	75
{3}	0	x_3	75
{1,2}	0	$x_1 + x_2$	125
{1,3}	0	$x_1 + x_3$	125
{2,3}	100	$x_2 + x_3 - 100$	50

The excess vector is $(50, 50, 75, 75, 125, 125)$.

Can we adjust the allocation to make one whose excess vector follows this one in the lexicographic ordering? If we take anything from Creditor 1, the first 50 in the table will fall, so the new excess vector will *precede* the old one in the lexicographic ordering. If we take anything from Creditor 2 or 3 and give it to Creditor 1, the other 50 in the table will fall, so the new excess vector will again precede the old one in the lexicographic ordering. The the only remaining possibility is to take something from Creditor 2 or 3 and give it to the other. If we do this, the two 50's in the table will remain, but one of the 75's will decrease. Again the new excess vector will precede the old one in the lexicographic ordering. We conclude that the allocation $(50, 75, 75)$ proposed by the Mishna is the nucleolus of the associated cooperative game.

Theorem 6.3 (Aumann-Maschler)**.** *In any estate division problem, the unique allocation that is consistent with the Contested Garment Rule is also the nucleolus of the associated cooperative game.*

It is a remarkable fact that Aumann and Maschler discovered the relationship between the Talmud's proposed allocations and the nucleolus before they thought of the relation to the Contested Garment Rule. Here is the story as told by Aumann in [2]: "Mike and I sat down to try to figure out what is going on in that passage. We put the nine relevant numbers on the blackboard in tabular form and gazed at them mutely. There seemed no rhyme or reason to them—not equal, not proportional, nothing. We tried the Shapley value of the corresponding coalitional game; this, too, did not work. Finally one of us said, let's try the nucleolus; to which the other responded, come on, that's crazy, the nucleolus is an extremely sophisticated notion of modern mathematical game theory, there's no way that the sages of the Talmud could possibly have thought of it. What do you care, said the first; it will cost us just fifteen minutes of calculation. So we did the calculation, and the nine numbers came out precisely as in the Talmud!" They then discovered by a literature search that the nucleolus had recently been proved to have a consistency property: if you look at the amounts assigned by the nucleolus to a subset of players, this is precisely the nucleolus of the reduced game with only those players and value equal to the total assigned to them by the nucleolus of the original game.

7. Final remark

There is one aspect of the Aumann-Maschler solution that bothered me. Their solution has nothing whatever to do with proportional division. Nevertheless, among the three examples of estate division given in the Talmud, one, the last, is proportional: each creditor gets exactly half her claim. Was the Talmud trying to lead us astray?

Joseph Bak of the City College of New York saw the slides for a talk I had given in which I asked this question. He sent the following answer: in any estate division problem, if the estate is exactly half the total of the debts, then the Aumann-Maschler rule will produce proportional division. In fact each debtor will get exactly half what she is owed.

This is easy to see from Figure 5.2. The volumes of the bottoms of all the glasses add up to $\frac{1}{2}(d_1 + \cdots + d_n)$. If this is the amount of the estate, it will exactly fill all the bottoms. Thus the ith debtor gets $\frac{d_i}{2}$.

If one gives examples of the Aumann-Maschler estate division rule using nice round numbers like 100, 200, etc. for the estate and the debts, it is quite easy for one of the examples to have the estate equal to exactly half the total debts, thus producing proportional division. This may be what happened in the Talmud.

References

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