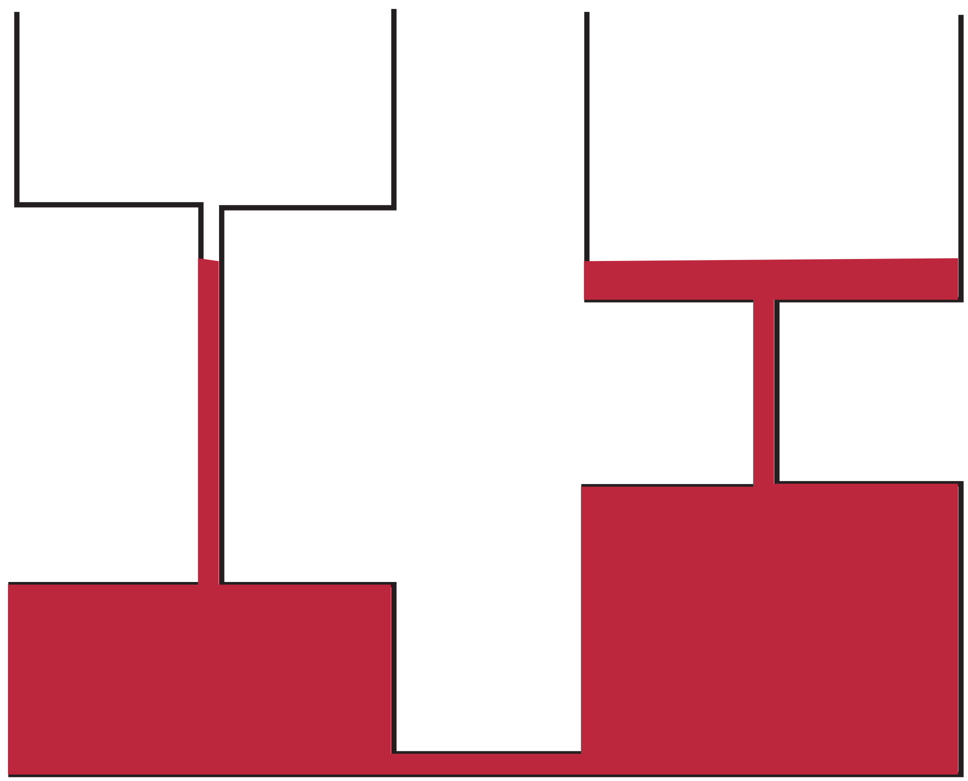
How the Talmud Divides an Estate Among Creditors



Steve Schecter
North Carolina State University

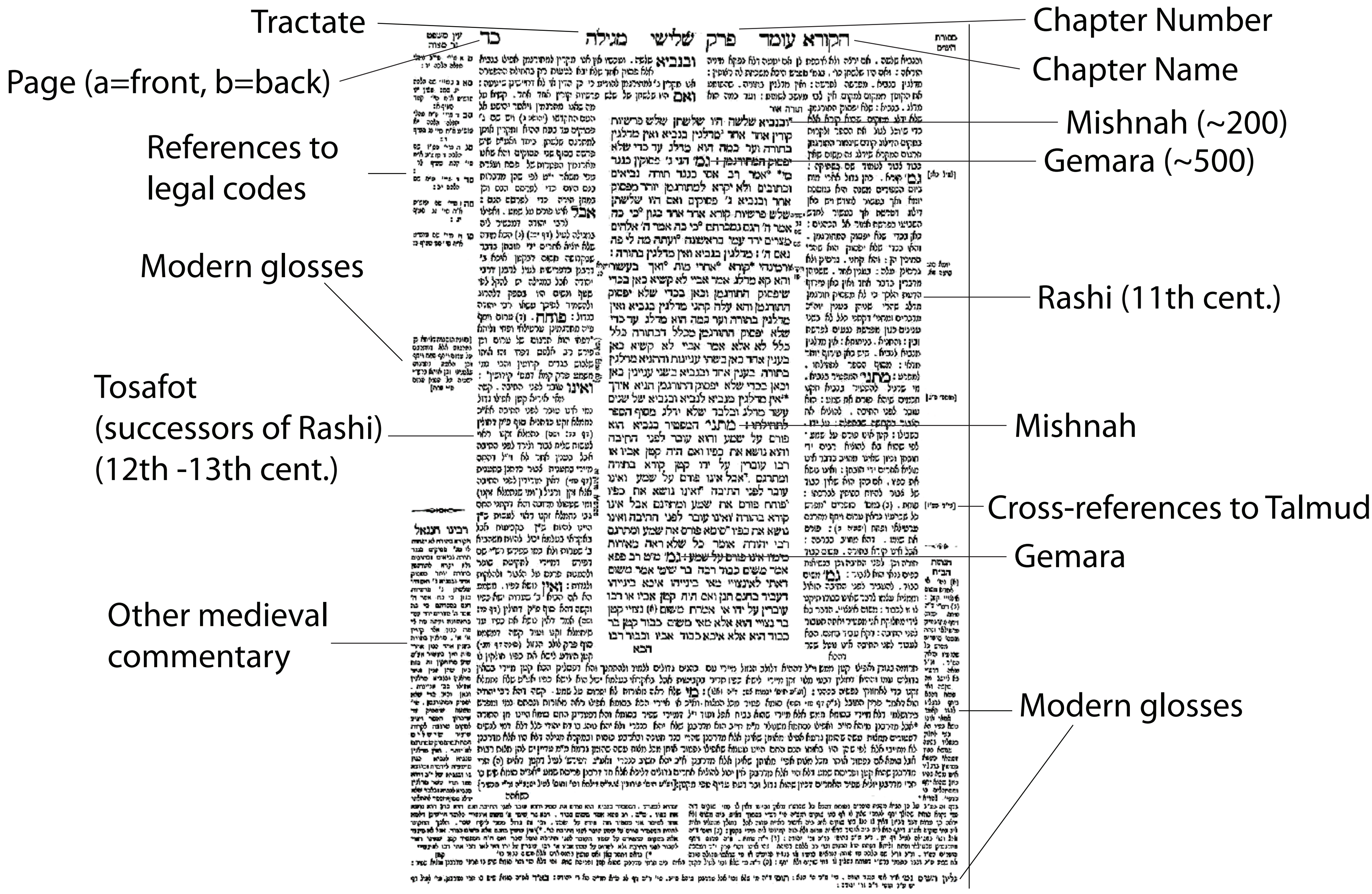
The Talmud consists of . . .

- Mishna (c. 200 AD), a written compendium of Judaism's Oral Law.

- Gemara (c. 500 AD), a record of discussion by rabbis about the Mishna.

First printed in Italy around 1520.

Today's printings: 60 tractates in 20 volumes occupying one meter of shelf space.


Tractate
Chapter Number
Page (a=front, b=back)
Chapter Name
Mishnah (~200)
References to legal codes
Gemara (~500)
Modern glosses
Rashi (11th cent.)
Tosafot (successors of Rashi) (12th -13th cent.)
Mishnah
Cross-references to Talmud
Gemara
Other medieval commentary
Modern glosses

A Problem from the Talmud

A man dies leaving

- an estate of size e;
- debts to Creditors $1, \ldots, n$ of $d_1, \ldots, d_n$;
- $e < d_1 + \cdots + d_n$.

How much should each creditor get?

A Mishna (Tractate Ketubot 93a): Assume $d_1 = 100$, $d_2 = 200$, $d_3 = 300$.

- If $e = 100$, each creditor gets 33 1/3.
- If $e = 200$, Creditor 1 gets 50, Creditors 2 and 3 get 75 each.
- If $e = 300$, Creditor 1 gets 50, Creditor 2 gets 100, creditor 3 gets 150.

A literature stretching across 1500 years deals with the question: what algorithm is this Mishna describing?

Of course, as in any legal system, the answer must be based on other Talmudic principles.

Ideas from the talmudic literature:

- The Mishna is wrong. (This is the majority view.)
- There are special circumstances that have not been explained.
- There is an (unconvincing) rational explanation.
- The text is corrupt.

Alfasi (11th century): “My predecessors discussed this Mishna and its Gemara at length, and were unable to make sense of it.”

Discussion based on ...

R. J. Aumann and M. Maschler, “Game Theoretic Analysis of a Bankruptcy Problem from the Talmud,” J. Economic Theory 36 (1985), 195–213.

M. M. Kaminski, “ ‘Hydraulic’ rationing,” Mathematical Social Sciences 40 (2000), 131–155.

S., “How the Talmud Divides an Estate Among Creditors,” expository article in Bridging Mathematics, Statistics, Engineering and Technology: Contributions from the Seminar on Mathematical Sciences and Applications, Springer, 2012.

An *estate division problem* is a pair $(e,(d_1,\ldots,d_n))$ such that:

- $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n$.
- Let $d = d_1 + \ldots + d_n$. Then $0 < e < d$.

A *division* of the estate is $(x_1,\ldots,x_n)$ with $0 \leq x_i$ for all i and $x_1 + \cdots + x_n = e$.

What are some rational ways to divide an estate among creditors?

Proportional Division

- Assign to creditor i the amount $(d_i/d)e$.
- This method treats each dollar of debt as equally worthy.
- Mishna appears to use this idea when $e = 300$.
- Secular legal systems typically follow this idea.

Equal Division of Gains

- Assign to each creditor the amount e/n.
- This method treats each creditor as equally worthy.
- Mishna appears to use this idea when $e = 100$.
- Only sensible for small estates ($e/n \leq d_1$).

Constrained Equal Division of Gains

- Give each creditor the same amount, but don't give any creditor more than her claim.

- In other words, choose a number a such that
$$\min(d_1, a) + \min(d_2, a) + \ldots \min(d_n, a) = e.$$
Then assign to creditor i the amount $\min(d_i, a)$.

- The number a exists and is unique because for fixed $(d_1, \ldots, d_n)$, the left-hand side is a function of a that maps the interval $[0, d]$ onto itself and is strictly increasing on this interval.

- This rule was adopted by Maimonides (12th century).

- It is inconsistent with our Mishna (produces equal division in all our cases).

Equal Division of Losses

- Make each creditor take the same loss. The total loss to the creditors is $d - e$, so assign to creditor i the amount $d_i - (d - e)/n$.

- Only sensible for large estates (need $d_1 > (d - e)/n$).

Constrained Equal Division of Losses

- Make each creditor take the same loss, but don't make any creditor lose more than her claim.
- Perhaps used by Maimonides to deal with reneging in auctions.

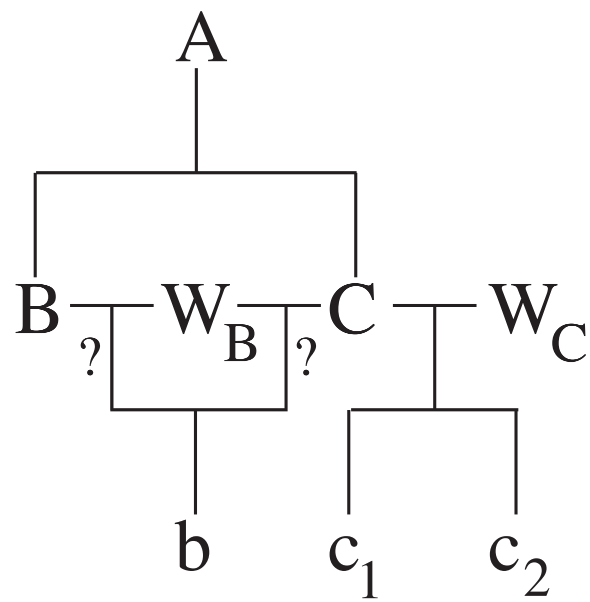
Some Other Mishnas

Tractate Baba Metzia 2a: "Two hold a garment; one claims it all, the other claims half. Then the one is awarded three-fourths, the other one-fourth."

Explanation (Rashi, 11th century): The one who claims half concedes that half belongs to the other. Therefore only half is in dispute. It is split equally.

Tractate Yevamot 38a:

- B dies childless.
- His widow marries his brother, C (levirate marriage).
- C already has two sons, c_1 and c_2, by his first wife.
- Eight months later the widow gives birth to a son, b, whose father is therefore doubtful.
- Next C dies.
- Finally, A, the father of B and C dies.
- Question: How is A's estate to be divided among his grandchildren b, c_1, and c_2?



b says: Half goes to A's son B and half to A's son C. I am B's only son, so I get his half. C's half should be divided between c_1 and c_2.

c_1 and c_2 say: B had no children, and C had three sons. Therefore the entire estate goes to C, and then is divided equally among the the three grandchildren.

The decision:

- c_1 and c_2 are treated as one claimant, b as another.
- The 1/2 of the estate that b concedes is not his goes to c_1 and c_2.
- The 1/3 of the estate that c_1 and c_2 concede is not theirs goes to b.
- The remainder of the estate, 1/6, is split equally: 1/12 to c_1 and c_2, 1/12 to b.
- Thus b gets $5/12$ of the estate, and c_1 and c_2 get 7/12 to split.

Neither Mishna treats a situation exactly analogous to an estate with creditors: there all claims are valid, in these two Mishnas both claims cannot be valid. Nevertheless, applied to an estate with two creditors, we get:

Notation: $a^+ = \max(a, 0)$.

Contested Garment Rule. Consider an estate division problem with two creditors: $0 \le d_1 \le d_2$, $d = d_1 + d_2$, $0 < e < d$.

- Creditor 2 concedes $(e - d_2)^+$ to Creditor 1.
- Creditor 1 concedes $(e - d_1)^+$ to Creditor 2.
- The remainder of the estate, $e - (e - d_1)^+ - (e - d_2)^+$, is divided equally.
- Thus Creditor 1 receives
$$(e - d_2)^+ + \frac{1}{2}\big(e - (e - d_1)^+ - (e - d_2)^+\big).$$
Creditor 2 receives
$$(e - d_1)^+ + \frac{1}{2}\big(e - (e - d_1)^+ - (e - d_2)^+\big).$$

Is the Contested Garment Rule relevant to our Mishna?

Aumann and Maschler's Observation

Back to our Mishna: $d_1 = 100$, $d_2 = 200$, $d_3 = 300$.

- If $e = 100$, each creditor gets 33 1/3.
- If $e = 200$, Creditor 1 gets 50, Creditors 2 and 3 get 75 each.
- If $e = 300$, Creditor 1 gets 50, Creditor 2 gets 100, creditor 3 gets 150.

Aumann and Maschler's observation: Each of these divisions is consistent with the Contested Garment Rule in the following sense:

If any two creditors use the Contested Garment Rule to split the amount they were jointly awarded, each will get the amount he was actually awarded.

In an estate division problem $(e, (d_1, \ldots, d_n))$, a division $(x_1, \ldots, x_n)$ of the estate is *consistent with the Contested Garment Rule* if, for each pair (i, j), (x_i, x_j) is exactly the division produced by the Contested Garment Rule applied to an estate of size $x_i + x_j$ with debts d_i and d_j.

Theorem (Aumann-Maschler). In any estate division problem, there is exactly one division of the estate that is consistent with the Contested Garment Rule.

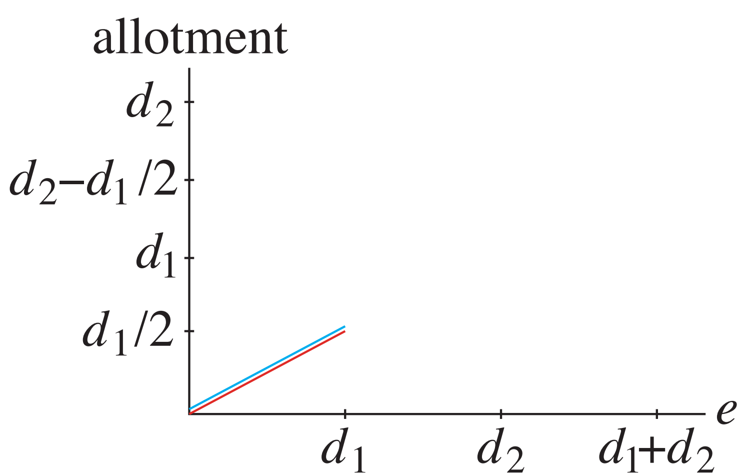
Consequences of Contested Garment Rule

Contested Garment Rule. Consider an estate division problem with two creditors: $0 \le d_1 \le d_2$, $d = d_1 + d_2$, $0 < e < d$. Creditor 2 concedes $(e - d_2)^+$ to Creditor 1. Creditor 1 concedes $(e - d_1)^+$ to Creditor 2. The remainder of the estate, $e - (e - d_1)^+ - (e - d_2)^+$, is divided equally.

(1) If $e \le d_1$, nothing is conceded, so everything is split:

$$\text{Creditor 1: } \frac{e}{2}.$$

$$\text{Creditor 2: } \frac{e}{2}.$$

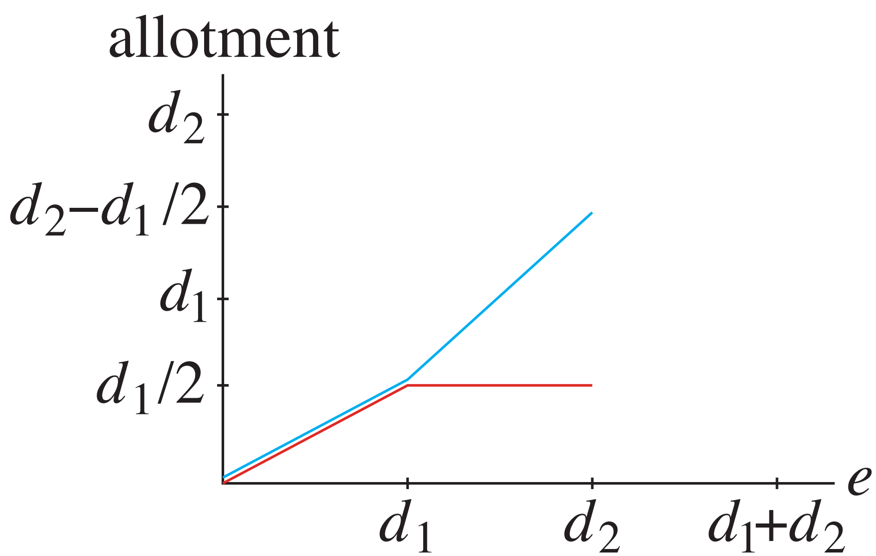


Each additional dollar of estate value produces an equal gain for each creditor.

(2) If $d_1 < e \leq d_2$, $e - d_1$ is conceded to Creditor 2, nothing is conceded to Creditor 1, and the remainder, d_1, is split.

Creditor 1: $\dfrac{d_1}{2}$.

Creditor 2: $(e - d_1) + \dfrac{d_1}{2}$.

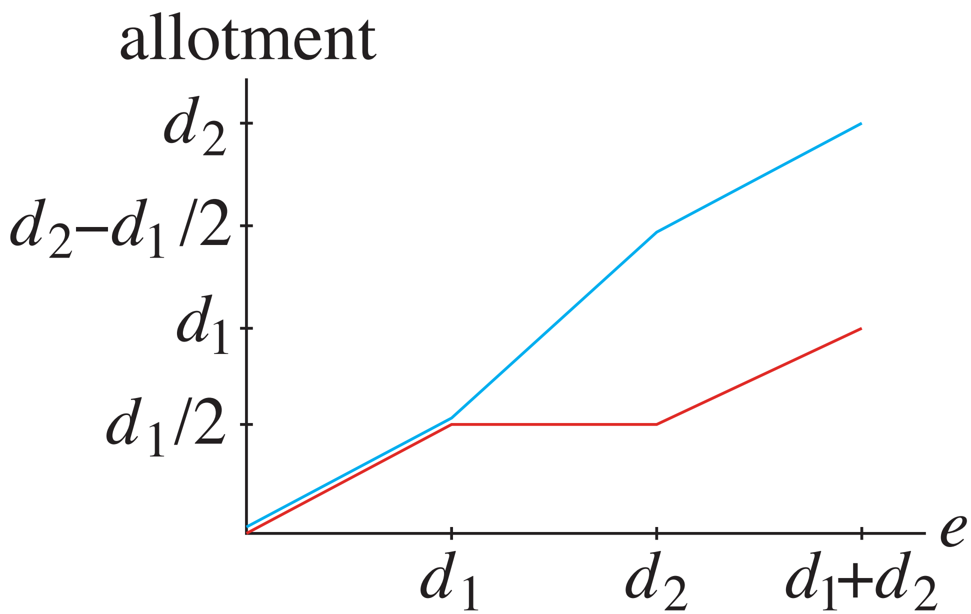


When $e = d_1$, the estate is split, each Creditor has a gain of $d_1/2$. Thereafter each additional dollar of estate value goes to Creditor 2. When e reaches d_2, Creditor 1 gets $d_1/2$ and Creditor 2 gets $d_2 - \frac{d_1}{2}$, so each Creditor has a loss of $d_1/2$. Previously Creditor 2's loss was larger.

(3) If $d_2 < e \leq d_1 + d_2$, $e - d_1$ is conceded to Creditor 2, $e - d_2$ is conceded to Creditor 1, and the remainder, $e - (e - d_1) - (e - d_2) = d_1 + d_2 - e$, is split.

Creditor 1: $e - d_2 + \frac{1}{2}(d_1 + d_2 - e) = d_1 - \frac{1}{2}(d_1 + d_2 - e)$.

Creditor 2: $e - d_1 + \frac{1}{2}(d_1 + d_2 - e) = d_2 - \frac{1}{2}(d_1 + d_2 - e)$.



The part of the estate above d_2 is split equally. The two creditors' losses remain equal.

Conclusion: The Contested Garment Rule linearly interpolates between Equal Division of Gains for $e \leq d_1$ and Equal Division of Losses for $d_2 \leq e$. At $e = d_1$, both creditors gain $d_1/2$; at $e = d_2$, both creditors lose $d_1/2$.

"More Than Half is Like the Whole"

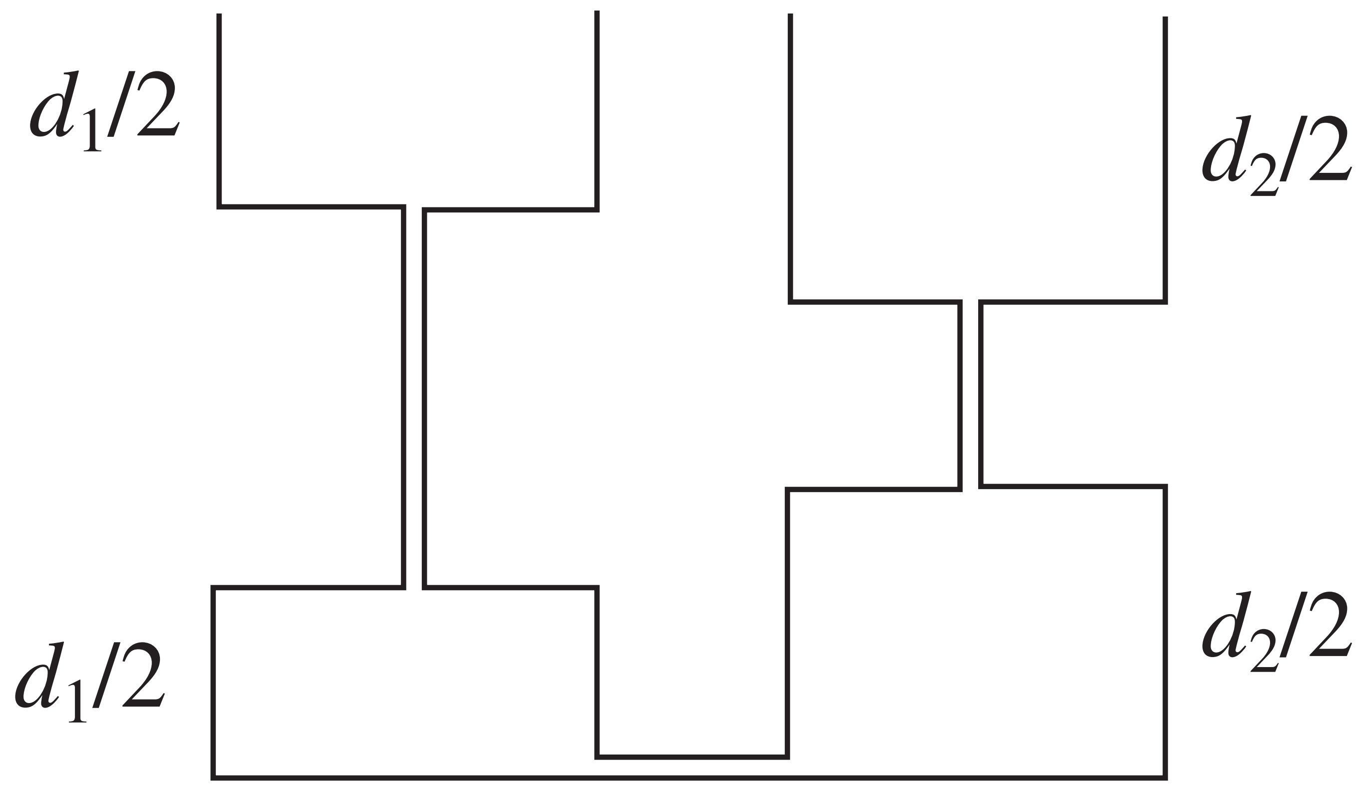
The Contested Garment Rule is perhaps related to the Talmudic principle that "more than half is like the whole."

Example: Normally a lender has an automatic lien on a borrower's real property. However, if the property is worth less than half the loan and the borrower defaults, the lender may not take the borrower's property (Arakhin 23b). Rashi explains: since the property is grossly inadequate to repay the loan, the loan is presumed to have been made "on trust," so the lender has no lien on the borrower's property.

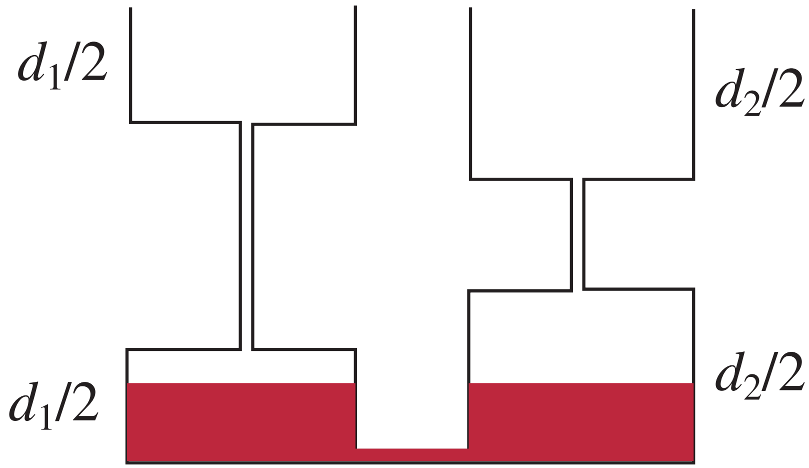
In other words: if the property is worth less than half the loan, you cannot rely on the loan's being repaid, so any repayment you do get is a gain relative to your expectation. If the property is worth more than half the loan, you expect the loan to be repaid, so any repayment you do not get is a loss relative to your expectation.

The Contested Garment Rule is perhaps a sophisticated alterative to the Talmudic principle that the dividing line between two approaches to a problem is at the number one-half.

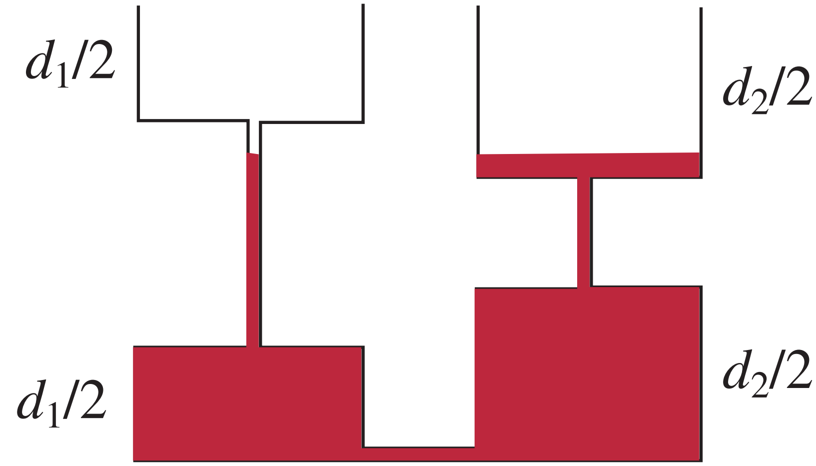
Interpreting the Contested Garment Rule in Glassware



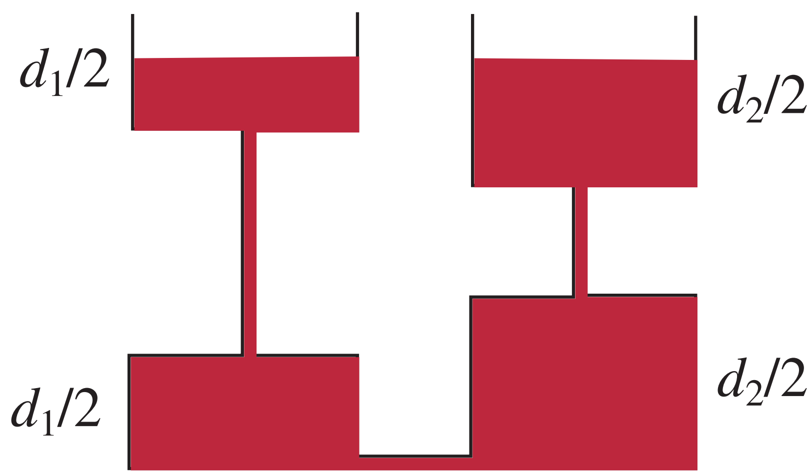
If an amount of liquid e is poured into this glassware, it will divide itself between the two creditors according to the Contested Garment Rule.



$e < d_1$



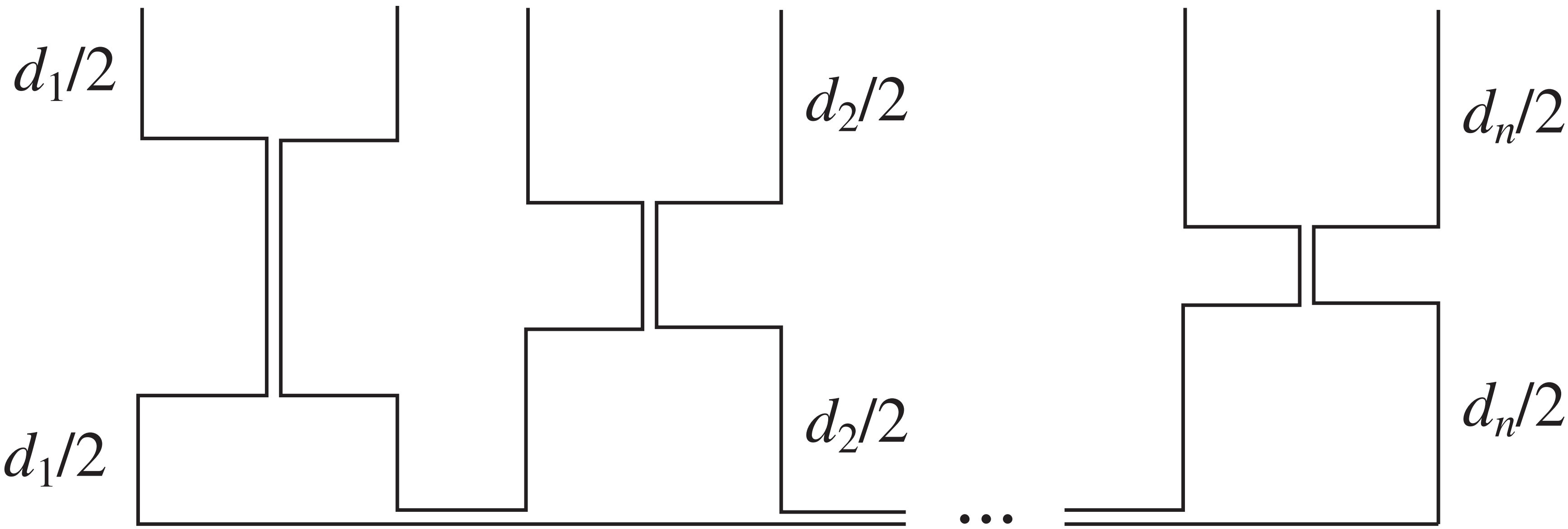
$d_1 < e < d_2$



$d_2 < e < d_1 + d_2$

Proof of the Aumann-Maschler Theorem

Given $d_1, \ldots, d_n$, construct the following glassware:



Pour in an amount e of liquid. It will divide itself among the n creditors in a way that is consistent with the Contested Garment Rule (since the glasses for each pair of creditors have the the same height of liquid). This division is unique: if we raise the height in one glass, we must raise the height in all, and the total amount of liquid will increase.

Marek Kaminski: “I learned about the bankruptcy problem from Peyton Young, a terrific teacher and scholar, at a class in fair allocation he taught at the University of Maryland, College Park. He patiently tried to explain to us his concept of parametric representation of allocation methods. Sitting in class, I was repeatedly failing to visualize the parametric representation of the Talmudic solution and, displeased with myself, I stopped listening and started thinking about an alternative. The “hydraulic” idea came to my mind—as it happens—in one of those unexplainable flashes. It seemed simple enough to work it out mathematically and it was applicable for practically all interesting methods. Later, I proved that in fact it is closely related to parametric representation.”

The Aumann-Maschler Theorem and Game Theory

A *cooperative game* consists of

- a set of players $\{1,\dots,n\}$.
- a value V to be distributed among the players.

Let S be a subset of $\{1,\dots,n\}$ (a *coalition*). S can get for itself an value $v(S)$ no matter what.

Assumptions:

- $v(\emptyset)=0$.
- $v(\{1,\dots,n\})=V$.
- If S_1 and S_2 are disjoint, then $v(S_1)+v(S_2)\le v(S_1\cup S_2)$.

An *allocation* is a vector $x=(x_1,\dots,x_n)$ such that all $x_i\ge 0$ and $x_1+\cdots+x_n=V$.

Problem: Choose the allocation.

Given an allocation x, the coalition S achieves the *excess* $e(x,S)=\sum_{j\in S}x_j-v(S)$.

Idea: Coalitions with low excess will complain that they have been treated unfairly and won't agree to the allocation. Choose x to minimize the complaints.

More precisely, given an allocation x, calculate all $2^n - 2$ excesses $e(x,S)$. (We ignore the empty set and the set $\{1,\ldots,n\}$.) Order them from smallest to largest to form an *excess vector* $e \in \mathbb{R}^{2^n-2}$.

Given two excess vectors we can ask which precedes which in the lexicographic ordering.

- Example: $(1,2,4,5)$ precedes $(2,2,2,6)$.
- Example: $(2,2,2,6)$ precedes $(2,2,3,5)$.

Definition. The *nucleolus* of a cooperative game is the allocation whose excess vector comes last in the lexicographic ordering.

Theorem. Every cooperative game has a unique nucleolus.

Example. A man dies leaving an estate of 200. There are three creditors with claims of 100, 200, and 300. Any coalition can guarantee itself whatever is left after those not in the coalition are paid in full.

The nucleolus for this game is $x = (x_1, x_2, x_3) = (50, 75, 75)$.

Explanation: The coalition $\{2,3\}$ can guarantee itself 100. No other coalition can guarantee itself anything. So . . .

If allocation is $x = (x_1, x_2, x_3) = (50, 75, 75)$:

S	$v(S)$	$e(x,S)$	$x = (50,75,75)$
$\{1\}$	0	x_1	50
$\{2\}$	0	x_2	75
$\{3\}$	0	x_3	75
$\{1,2\}$	0	$x_1 + x_2$	125
$\{1,3\}$	0	$x_1 + x_3$	125
$\{2,3\}$	100	$x_2 + x_3 - 100$	50

The excess vector is $(50, 50, 75, 75, 125, 125)$.

Can we adjust the allocation to make one whose excess vector follows this one in the lexicographic ordering?

- Take from Creditor 1: first 50 falls, so new excess vector *precedes* old.
- Give to Creditor 1: the other 50 falls, so new excess vector *precedes* old.
- Take from Creditor 2 or 3 and give it to the other: the two 50s remain, but one of the 75s falls, so new excess vector *precedes* old.

Conclusion: (50, 75, 75) is the nucleolus.

Theorem (Aumann-Maschler). In any estate division problem, the unique allocation that is consistent with the Contested Garment Rule is also the nucleolus of the associated cooperative game.

Robert Aumann: “Mike and I sat down to try to figure out what is going on in that passage. We put the nine relevant numbers on the blackboard in tabular form and gazed at them mutely. There seemed no rhyme or reason to them—not equal, not proportional, nothing. We tried the Shapley value of the corresponding coalitional game; this, too, did not work. Finally one of us said, let’s try the nucleolus; to which the other responded, come on, that’s crazy, the nucleolus is an extremely sophisticated notion of modern mathematical game theory, there’s no way that the sages of the Talmud could possibly have thought of it. What do you care, said the first; it will cost us just fifteen minutes of calculation. So we did the calculation, and the nine numbers came out precisely as in the Talmud!”

They then discovered by a literature search that the nucleolus had recently been proved to have a consistency property: amounts assigned by the nucleolus to a subset of players are precisely the nucleolus of the reduced game with only those players and value equal to the total assigned to them by the nucleolus of the original game.

Final Remark

Our Mishna gave the proportional allocation for one case.

Question. In an estate division problem with $(d_1, \ldots, d_n)$ fixed, is there always a value of e for which the division that is consistent with the Contested Garment Rule is also the proportional division?

Answer: Yes, $e = \frac{1}{2}d$. Everyone gets half their claim.