THE SUBHARMONIC MELNIKOV INTEGRAL

S. SCHECTER

Consider

$$\dot{x} = f(x) + \epsilon g(t, x, \epsilon), \quad x \in \mathbb{R}^2, \quad \epsilon \text{ small.}$$

We assume:

(1) $\dot{x} = f(x)$ has a one-parameter family of closed orbits with changing period.
(2) $g(t, x, \epsilon)$ has period $T(\epsilon)$.
(3) One of the closed orbits, Γ, with period T, is in resonance with g, i.e., there are positive integers n and m such that

$$T(\epsilon) = \frac{n}{m}T + k\epsilon + \mathcal{O}(\epsilon^2).$$

We ask the question: are there orbits of period

$$mT(\epsilon) = nT + mk\epsilon + \mathcal{O}(\epsilon^2)$$

near Γ?

Let

$$x = \phi(t, y, \epsilon),$$

denote the solution with $x(0) = y$ and parameter value ϵ. The Poincaré map of $\dot{x} = f(x) + \epsilon g(t, x, \epsilon)$ on $\mathbb{R}^2$ is

$$P(y, \epsilon) = \phi(T(\epsilon), y, \epsilon),$$

ie., we advance one period of the forcing function. Then

$$P^m(y, \epsilon) = \phi(mT(\epsilon), y, \epsilon) = \phi(nT + mk\epsilon + \mathcal{O}(\epsilon^2), y, \epsilon).$$

The displacement $d(y, \epsilon)$ is defined by

$$d(y, \epsilon) = P^m(y, \epsilon) - y = \phi(nT + mk\epsilon + \mathcal{O}(\epsilon^2), y, \epsilon).$$

We seek solutions of the equation $d(y, \epsilon) = 0$. Notice that $d(y, \epsilon) \in \mathbb{R}^2$. Unlike the situations we have previously considered, we do *not* have $d(y, 0) \equiv 0$, because the closed orbits of $\dot{x} = f(x)$ near Γ have periods different from T.

Let y_0 be a point on Γ. We will need a convenient coordinate system near y_0. Let Σ be a line segment through y_0 that is perpendicular to Γ. Let

$$v_0 = \begin{pmatrix} -f_2(y_0) \\ f_1(y_0) \end{pmatrix} / \|f(y_0)\|.$$

The vector v_0 is a unit vector that is perpendicular to $f(y_0)$, so it is perpendicular at y_0 to Γ. Let

$$s(z_2) = y_0 + z_2 v_0.$$

Then z_2 parameterizes Σ.

To simplify notation, let

$$\omega(t, y) = \phi(t, y, 0).$$

Date: April 15, 2013.

With this notation, the coordinate system we need is

$$y = H(z) = \omega(z_1, y_0 + z_2 v_0), \quad \|z\| \text{ small.}$$

We define

$$\tilde{\phi}(t, z, \epsilon) = H^{-1}\phi(t, H(z), \epsilon),$$
$$\tilde{P}(z, \epsilon) = H^{-1}P(H(z), \epsilon) = H^{-1}\phi(T(\epsilon), H(z), \epsilon) = \tilde{\phi}(T(\epsilon), z, \epsilon).$$

$\tilde{\phi}$ and $\tilde{P}$ are the flow and Poincaré map in z-coordinates. Notice that $\tilde{\phi}(t, z, \epsilon)$ is only defined when $\|z\|$ is small and $\phi(t, H(z), \epsilon)$ is near y_0. We calculate:

$$\tilde{P}^m(z, 0) = H^{-1}\phi\big(mT(0), H(z), 0\big) = H^{-1}\omega\big(mT(0), H(z)\big)$$
$$= H^{-1}\omega\big(nT, \omega(z_1, s(z_2))\big) = H^{-1}\omega\big(nT + z_1, s(z_2)\big).$$

Denote the the period of the orbit of $\dot{x} = f(x)$ through $s(z_2)$ by

$$\tau(z_2) = T + \sigma(z_2) \text{ with } \sigma(0) = 0.$$

We assume:

$$\tau'(z_2) = \sigma'(z_2) \neq 0.$$

In other words, the period of the closed orbits of $\dot{x} = f(x)$ near Γ is changing in a nondegenerate manner near Γ. Then

$$\tilde{P}^m((z_1, z_2), 0) = H^{-1}\omega\big(nT + z_1, s(z_2)\big) = H^{-1}\omega\big(n(\tau(z_2) - \sigma(z_2)) + z_1, s(z_2)\big)$$
$$= H^{-1}\omega\big(-n\sigma(z_2) + z_1, \omega(n\tau(z_2), s(z_2))\big) = H^{-1}\omega\big(-n\sigma(z_2) + z_1, s(z_2)\big) = \big(-n\sigma(z_2) + z_1, z_2\big).$$

Summarizing:

$$\tilde{P}^m\big((z_1, z_2), 0\big) = \big(-n\sigma(z_2) + z_1, z_2\big).$$

Instead of solving $d(y, \epsilon) = 0$, we can instead define

$$\tilde{d}(z, \epsilon) = H^{-1}d(H(z), \epsilon) = \tilde{P}^m(z, \epsilon) - z,$$

and try to solve $\tilde{d}(z, \epsilon) = 0$.

We have

$$\tilde{d}((z_1, z_2), 0) = \tilde{P}^m((z_1, z_2), 0) - (z_1, z_2) = \big(-n\sigma(z_2) + z_1, z_2\big) - (z_1, z_2) = \big(-n\sigma(z_2), 0\big).$$

Notice that

$$\tilde{d}((z_1, 0), 0) \equiv 0,$$

and that

$$D_{(z_1, z_2)}\tilde{d}((z_1, 0), 0) = \begin{pmatrix} 0 & -n\sigma'(0) \\ 0 & 0 \end{pmatrix}.$$

From the assumption that $\sigma'(0) \neq 0$, we see that $\frac{\partial \tilde{d}_1}{\partial z_2}((0, 0), 0) \neq 0$.

Because the matrix $D_{(z_1, z_2)}\tilde{d}((z_1, 0), 0)$ is not invertible, we cannot use the Implicit Function Theorem to solve the equation $\tilde{d}(z, \epsilon) = 0$ for z as a function of ϵ. However, because $\frac{\partial \tilde{d}_1}{\partial z_2}((0, 0), 0) \neq 0$, we *can* use the Implicit Function Theorem to solve the equation $\tilde{d}_1((z_1, z_2), \epsilon) = 0$ for z_2 as a function of (z_1, ϵ). We can then substitute the solution $z_2 = z_2(z_1, \epsilon)$ into the remaining equation $\tilde{d}_2((z_1, z_2), \epsilon) = 0$, which will leave us just one equation in two variables to solve. This is an improvement: originally we had two equations in three variables.

Believe it or not, this simple idea has a name: Lyapunov-Schmidt reduction. See our text, p. 224, where Lyapunov-Schmidt reduction is used instead of center manifold reduction to study bifurcation of equilibria. The disadvantage of Lyapunov-Schmidt reduction for studying bifurcation of equilibria is that it finds the equilibria but does not directly give information about the flow near the equilibria. You can look up the general version of Lyapunov-Schmidt reduction in Wikipedia.

Since $\tilde{d}_1((z_1,0),0) \equiv 0$, when we solve $\tilde{d}_1((z_1,z_2),\epsilon) = 0$ for z_2 as a function of (z_1,ϵ), we will get

$$z_2 = a(z_1,\epsilon) = \epsilon b(z_1,\epsilon).$$

Substituting into $\tilde{d}_2((z_1,z_2),\epsilon) = 0$, we obtain

$$\tilde{d}_2((z_1,\epsilon b(z_1,\epsilon)),\epsilon) = \epsilon c(z_1,\epsilon) = 0. \tag{1}$$

The reason $\tilde{d}_2((z_1,\epsilon b(z_1,\epsilon)),\epsilon) = \epsilon c(z_1,\epsilon)$ is that $\tilde{d}_2((z_1,0),0) \equiv 0$.

From (1) and the definition of $\tilde{d}$, we have

$$\tilde{d}_2((z_1,\epsilon b),\epsilon) = \tilde{P}_2^m((z_1,\epsilon b),\epsilon) - \epsilon b = \tilde{\phi}_2(mT(\epsilon),(z_1,\epsilon b),\epsilon) - \epsilon b.$$

($\tilde{\phi}_2$ is the second coordinate of $\tilde{\phi}$.) Differentiate both sides of the equation

$$\tilde{\phi}_2(mT(\epsilon),(z_1,\epsilon b),\epsilon) - \epsilon b = \epsilon c$$

with respect to ϵ and set $\epsilon = 0$:

$$\frac{\partial\tilde{\phi}_2}{\partial t}(mT(0),(z_1,0),0)mk + \frac{\partial\tilde{\phi}_2}{\partial z_2}(mT(0),(z_1,0),0)b + \frac{\partial\tilde{\phi}_2}{\partial \epsilon}(mT(0),(z_1,0),0) - b = c,$$

where b and c are evaluated at $(z_1,0)$. Since $mT(0) = nT$, $\frac{\partial\tilde{\phi}_2}{\partial t} = 0$ when $\epsilon = 0$, and $\frac{\partial\tilde{\phi}_2}{\partial z_2} = 1$ when $\epsilon = 0$, this simplifies to

$$\frac{\partial\tilde{\phi}_2}{\partial \epsilon}(nT,(z_1,0),0) = c(z_1,0). \tag{2}$$

To compute $c(z_1,0)$, let $z_2 = 0$ and define $\tilde{z}_1(z_1,\epsilon)$ and $\tilde{z}_2(z_1,\epsilon)$ by

$$\tilde{\phi}(nT,(z_1,0),\epsilon) = (\tilde{z}_1(z_1,\epsilon),\tilde{z}_2(z_1,\epsilon)).$$

Then

$$\phi(nT,\omega(z_1,y_0),\epsilon) = \omega(\tilde{z}_1(z_1,\epsilon),s(\tilde{z}_2(z_1,\epsilon))). \tag{3}$$

Since

$$\tilde{\phi}(nT,(z_1,0),0) = (z_1,0),$$

we have

$$\tilde{z}_1(z_1,0) = z_1 \text{ and } \tilde{z}_2(z_1,0) = 0. \tag{4}$$

From (3) and (4),

$$\frac{\partial\phi}{\partial\epsilon}(nT,\omega(z_1,y_0),0) = \frac{\partial\omega}{\partial t}(z_1,y_0)\frac{\partial\tilde{z}_1}{\partial\epsilon}(z_1,0) + D_y\omega(z_1,y_0)\frac{\partial\tilde{z}_2}{\partial\epsilon}(z_1,0)v_0. \tag{5}$$

Consider the linear differential equation

$$\dot{X} = Df(\omega(t,y))X. \tag{6}$$

It has the vector solution $\frac{\partial\omega}{\partial t}(t,y)$ and the matrix solution $D_y\omega(t,y)$. Let $\Phi(t,s,y)$ denote the propagator of (6). Then $\Phi(t,0,y) = D_y\omega(t,y)$.

The adjoint equation of $\dot{X} = Df(\omega(t, y_0))X$ is

$$\dot{w} = -wDf(\omega(t, y_0)).$$

It has a solution $\psi(t)$ with $\psi(nT) = v_0^\top$.

Define the *subharmonic Melnikov integral*

$$M(z_1) = \int_0^{nT} \psi(\sigma) g(\sigma - z_1, \omega(\sigma, y_0), 0)\, d\sigma.$$

Theorem 1. *If $M(z_1) = 0$ and $M'(z_1) \neq 0$, then a curve of (m, n)-subharmonic solutions emerges from $\omega(z_1, y_0)$.*

To prove this theorem, we multiply (5) by $\psi(nT + z_1)$ and obtain

$$\begin{aligned}\psi(nT + z_1)&\frac{\partial \phi}{\partial \epsilon}(nT, \omega(z_1, y_0), 0) \\ &= \psi(nT + z_1)\frac{\partial \omega}{\partial t}(z_1, y_0)\frac{\partial \tilde{z}_1}{\partial \epsilon}(z_1, 0) + \psi(nT + z_1) D_y\omega(z_1, y_0)\frac{\partial \tilde{z}_2}{\partial \epsilon}(z_1, 0)v_0. \quad (7)\end{aligned}$$

Now

$$\frac{\partial \omega}{\partial t}(z_1, y_0) = \frac{\partial \omega}{\partial t}(nT + z_1, y_0),$$

so

$$\psi(nT+z_1)\frac{\partial \omega}{\partial t}(z_1, y_0) = \psi(nT+z_1)\frac{\partial \omega}{\partial t}(nT+z_1, y_0) = \psi(nT)\frac{\partial \omega}{\partial t}(nT, y_0) = v_0^\top f(y_0) = 0. \quad (8)$$

Also,

$$\psi(nT+z_1)D_y\omega(z_1, y_0) = \psi(nT+z_1)\Phi(z_1, 0, y_0) = \psi(nT+z_1)\Phi(nT+z_1, nT, y_0) = \psi(nT) = v_0^T,$$

so

$$\psi(nT + z_1)D_y\omega(z_1, y_0)\frac{\partial \tilde{z}_2}{\partial \epsilon}(z_1, 0)v_0 = v_0^\top \frac{\partial \tilde{z}_2}{\partial \epsilon}(z_1, 0)v_0 = \frac{\partial \tilde{z}_2}{\partial \epsilon}(z_1, 0). \quad (9)$$

From (8) and (9), (7) simplifies to

$$\psi(nT + z_1)\frac{\partial \phi}{\partial \epsilon}(nT, \omega(z_1, y_0), 0) = \frac{\partial \tilde{z}_2}{\partial \epsilon}(z_1, 0) = c(z_1, 0).$$

Now

$$\begin{aligned}\frac{\partial \phi}{\partial \epsilon}(nT, \omega(z_1, y_0), 0) &= \int_0^{nT} \Phi(nT, s, \omega(z_1, y_0))g(s, \omega(s + z_1, y_0), 0)\, ds \\ &= \int_0^{nT} \Phi(nT + z_1, s + z_1, y_0)g(s, \omega(s + z_1, y_0), 0)\, ds.\end{aligned}$$

Therefore

$$
\begin{aligned}
c(z_1,0) &= \psi(nT+z_1)\frac{\partial\phi}{\partial\epsilon}(nT,\omega(z_1,y_0),0)\\
&= \psi(nT+z_1)\int_0^{nT}\Phi(nT+z_1,s+z_1,y_0)g(s,\omega(s+z_1,y_0),0)\,ds\\
&= \int_0^{nT}\psi(s+z_1)g(s,\omega(s+z_1,y_0),0)\,ds = \int_{z_1}^{nT+z_1}\psi(\sigma)g(\sigma-z_1,\phi(\sigma,y_0),0)\,d\sigma\\
&= \int_0^{nT}\psi(\sigma)g(\sigma-z_1,\omega(\sigma,y_0),0)\,d\sigma.
\end{aligned}
$$

The theorem follows from this calculation. However, the last equality requires comment.

It is clearly correct if the integrand has period nT in σ. $g(\sigma-z_1,\omega(\sigma,y_0),0)$ has period $mT(0)=nT$. Up to a scalar multiple,

$$\psi(\sigma)=\exp\left(-\int_0^\sigma \operatorname{div} f(\omega(r,y_0))\,dr\right)\begin{pmatrix}-f_2(\omega(\sigma,y_0)) & f_1(\omega(\sigma,y_0))\end{pmatrix}.$$

The row vector $\begin{pmatrix}-f_2(\omega(\sigma,y_0)) & f_1(\omega(\sigma,y_0))\end{pmatrix}$ has period T in σ. From Liouville's formula, we know that

$$\det D_y\omega(T,y_0)=\exp\left(\int_0^T \operatorname{div} f(\omega(r,y_0))\,dr\right).$$

But $D_y\omega(T,y_0)$ is similar to

$$D_z\tilde{\phi}(T,0,0)=\begin{pmatrix}1 & -\sigma'(0)\\ 0 & 1\end{pmatrix}.$$

Since the second matrix has determinant 1, so does the first. Therefore the function

$$\exp\left(\int_0^\sigma \operatorname{div} f(\omega(r,y_0))\,dr\right),$$

which equals 1 at $\sigma=0$, also equals 1 at $\sigma=T$. Hence its inverse, the function

$$\exp\left(-\int_0^\sigma \operatorname{div} f(\omega(r,y_0))\,dr\right),$$

also equals 1 at $\sigma=0$ and at $\sigma=T$. Therefore it has period T in σ. It follows that $\psi(\sigma)$ has period T in σ.