MA 732 Test 2

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1. Consider the differential equation

$$\begin{aligned} \dot{x} &= 2xy \\ \dot{y} &= y^2 - x^2 \end{aligned}$$

   In polar coordinates, after dividing by r, this system becomes

$$\begin{aligned} \dot{r} &= r\sin\theta \\ \dot{\theta} &= -\cos\theta \end{aligned}$$

   (a) Find the equilibria of the polar system with $r = 0$.

   (b) Find the matrix of the linearization of the polar system at each of these equilibria and determine their type.

   (c) Use your analysis to draw the phase portrait of the original system near the origin. You should start by drawing the phase portrait of the polar system near the circle $r = 0$.

2. Consider the differential equation

$$\dot{x} = f(x, \mu) = xa(x, \mu)$$

   where $a(x, \mu)$ is at least C^2 with $a(0, 0) = 0$, $a_x(0, 0) = A \neq 0$, $a_\mu(0, 0) = 0$, and $a_{\mu\mu}(0, 0) = 2D < 0$. Thus we can write

$$a(x, \mu) = 0 + Ax + 0 \cdot \mu + Bx^2 + Cx\mu + D\mu^2 + \ldots$$

   with $A \neq 0$ and $D < 0$. This is not one of the bifurcations we have studied.

   (a) Near $(x, \mu) = (0, 0)$, all equilibria lie on one of two curves: $x = 0$ or $x = k(\mu)$ with $k(0) = 0$. Explain briefly

   (b) Let $k(\mu) = 0 + b\mu + c\mu^2 + \ldots$. Find b and c.

   (c) For small $\mu \neq 0$, is $x = 0$ an attractor or a repeller? To answer this question, you should look at $f_x(0, \mu)$.

3. Consider the differential equation

$$\begin{aligned} \dot{x} &= 2x - y - x^2 \\ \dot{y} &= \mu - 2x + y \end{aligned}$$

Notice that for $\mu = 0$ there is an equilibrium at $(x, y) = (0, 0)$. The linearization of the differential equation at this equilibrium has the matrix

$$\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$$

This matrix has eigenvalues 0 and 3. Corresponding eigenvectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(a) There is a center manifold of the form $y = h(x, \mu) = Ax + B\mu + Cx^2 + \dots$. Find A, B, and C.

(b) Use your answer to part (a) to find the first few terms in the Taylor series for the differential equation on the center manifold.

(c) Use your answer to part (b) to sketch the flow on the center manifold near $(x, \mu) = (0, 0)$.

(d) Describe the flow of the full system near $(x, y) = (0, 0)$ for small $\mu < 0$ and for small $\mu > 0$.