MA 732 Test 1

S. Schecter

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1. Consider the 2π-periodic differential equation

$$\dot{x} = \cos t + x \sin t - x^3.$$

   (a) Show that this differential equation has at least one 2π-periodic solution. (Suggestion: consider $\dot{x}$ for x very positive and for x very negative.)

   (b) Show that any 2π-periodic solution is attracting. (Calculate the derivative of the Poincaré map.)

   (c) Use parts (a) and (b) to explain why this differential equation has a *unique* 2π-periodic solution.

2. Define $A : C^0([a,b],\mathbb{R}) \to C^0([a,b],\mathbb{R})$ by

$$A\phi(t) = \int_a^t \phi(s)\,ds.$$

   A is a linear map. (You don't have to show this.)

   (a) Show that A is bounded.

   (b) Find $\|A\|$ and justify your answer.

3. Use center manifold reduction to draw the phase portrait near the origin for the system

$$\begin{aligned}\dot{x} &= y - x^2,\\ \dot{y} &= -2y + 2x^2 - 2xy.\end{aligned}$$

   Use the following facts: the linearization at the origin has eigenvalues 0 and -2, with corresponding eigenvectors

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\-2\end{pmatrix}.$$

   Therefore the center manifold has the form $y = Ax^2 + Bx^3 + \ldots$. You will need both A and B.

4. Let B be an invertible $n \times n$ matrix, and let $G : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 with $G(0) = 0$ and $DG(0) = 0$. We want to prove that if $\|y\|$ is small then the equation $Bx + G(x) = y$ has a unique small solution x (there could be other solutions with $\|x\|$ large), and that x is a C^1 function of y. More precisely, we want to prove:

If $\epsilon > 0$ is sufficiently small, then there exists $\delta > 0$ and a map

$$H : \{y \in \mathbb{R}^n : \|y\| < \delta\} \to \{x \in \mathbb{R}^n : \|x\| \leq \epsilon\}$$

such that (1) $x = H(y)$ is the only solution with $\|x\| \leq \epsilon$ of the equation $Bx+G(x) = y$ and (2) H is C^1.

Note that $Bx + G(x) = y$ if and only if $x = B^{-1}(y - G(x))$. Define $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $T(x, y) = B^{-1}(y - G(x))$. For a given $y \in \mathbb{R}^n$, $Bx + G(x) = y$ if and only if $x = T(x, y)$, i.e., if and only if x is a fixed point of $T(\cdot, y)$.

(a) First we choose $\epsilon > 0$ small enough so that if $\|x\| \leq \epsilon$ then $\|DG(x)\| \leq \frac{1}{2\|B^{-1}\|}$. Why can we do this?

(b) Next we let $\delta = \frac{\epsilon}{2\|B^{-1}\|}$. Show that if $\|y\| < \delta$, then T maps $\{x : \|x\| \leq \epsilon\}$ into itself.

(c) Show that for each y with $\|y\| < \delta$, T is a contraction of $\{x : \|x\| \leq \epsilon\}$.

(d) Let $H(y)$ be the fixed point of $T(\cdot, y)$. Explain why H has the required properties.