MA 732 Final Exam

S. Schecter

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1. Consider the 2π-periodic differential equation

$$\dot{x} = (1 + \cos t)x - x^2.$$

   (a) One solution is $x(t) \equiv 0$, so 0 is a fixed point of the Poincaré map $P(y)$. Show that this fixed point is repelling by calculating $P'(0)$.

   (b) Show that for $y > 2$, $P(y) < y$.

   (c) Use parts (a) and (b) to show that there is a 2π-periodic solution $x(t)$ with $0 < x(t) < 2$ for all t.

2. Let $B : [a, b] \to m \times n$ matrices be continuous, i.e., for each t in $[a, b]$, $B(t)$ is an $m \times n$ matrix, and $B(t)$ depends continuously on t. Define $A : C^0([a, b], \mathbb{R}^n) \to C^0([a, b], \mathbb{R}^m)$ by

$$A\phi(t) = \int_a^t B(s)\phi(s)\, ds.$$

   A is a linear map. (You don't have to show this.) Show that A is bounded.

3. Define $F : C^0([a, b], \mathbb{R}^n) \to \mathbb{R}$ by $F(\phi) = \int_a^b \phi(t) \cdot \phi(t)\, dt$. *Using the definition of derivative*, prove that F is differentiable with $DF(\phi)\psi = \int_a^b 2\phi(t) \cdot \psi(t)\, dt$. (You may assume that this formula, for a fixed ϕ, defines a bounded linear map from $C^0([a, b], \mathbb{R}^n)$ to $\mathbb{R}$. You may need the Cauchy-Schwartz inequality $|v \cdot w| \leq \|v\|\|w\|$.)

4. Consider the differential equation

$$\begin{aligned} \dot{x} &= 2xy \\ \dot{y} &= x^2 + y^2 \end{aligned}$$

   In polar coordinates, after dividing by r, this system becomes

$$\begin{aligned} \dot{r} &= r \sin\theta(3\cos^2\theta + \sin^2\theta) \\ \dot{\theta} &= \cos\theta(\cos^2\theta - \sin^2\theta) \end{aligned}$$

   (a) Find the equilibria of the polar system with $r = 0$. (There are six.)

(b) Find the signs of the eigenvalues of the linearization of the polar system at each of these equilibria.

(c) Use your analysis to draw the phase portrait of the original system near the origin. You should start by drawing the phase portrait of the polar system near the circle $r = 0$.

5. Consider the differential equation

$$\dot{x} = f(x, \mu)$$

with $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, and f at least C^3. Assume $f(0,0) = f_x(0,0) = f_{xx}(0,0) = 0$, $f_\mu(0,0) = A \neq 0$, $f_{xxx}(0,0) = 6D < 0$. Then we can write

$$f(x, \mu) = 0 + 0 \cdot x + A\mu + 0 \cdot x^2 + Bx\mu + C\mu^2 + Dx^3 + \dots$$

with $A \neq 0$ and $D < 0$. This is not one of the bifurcations we have studied.

(a) Near $(x, \mu) = (0, 0)$, all equilibria lie a unique curve $\mu = k(x)$ with $k(0) = 0$. Explain briefly.

(b) Let $k(x) = 0 + ax + bx^2 + cx^3 + \dots$. Find a, b, and c.

(c) Are small $x \neq 0$ attractors or a repellers? To answer this question, you should look at $f_x(x, k(x))$.

6. Consider the differential equation

$$\begin{aligned} \dot{x} &= y + \mu x + x^2, \\ \dot{y} &= y + xy + 2x^2. \end{aligned}$$

Notice that $(x, y) = (0, 0)$ is an equilibrium for all values of μ. For $\mu = 0$ the linearization of the differential equation at this equilibrium has the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues of this matrix are 0 and 1. Corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) Since $(x, y) = (0, 0)$ is an equilibrium for all values of μ, the center manifold can be expressed as $y = h(x, \mu) = x(A + Bx + C\mu + \dots)$. Find A, B, and C.

(b) Show that on the center manifold, a transcritical bifurcation occurs at $\mu = 0$.

(c) Use your answer to part (b) to sketch the flow on the center manifold near $(x, \mu) = (0, 0)$.

(d) Describe the flow of the full system near $(x, y) = (0, 0)$ for small $\mu < 0$ and for small $\mu > 0$.