MA 732 S. Schecter

Test 1 Answers

(Thanks to Alison Margolskee!)

1. $\dot{x} = \cos t + x \sin t - x^3$

(a) There is at least one 2π-periodic sol'n

For $x >> 0$, $\dot{x} < 0$

$\Rightarrow$ for $x_1 >> 0$, $P(x_1) = \phi(2\pi, x_1) < x_1$

For $x << 0$, $\dot{x} > 0$

$\Rightarrow$ for $x_2 << 0$, $P(x_2) = \phi(2\pi, x_2) > x_2$

By continuity of $P(x)$, $\exists\, x_0$, $x_2 < x_0 < x_1$

s.t. $P(x_0) = x_0$, $\phi(t, x_0)$ is 2π periodic

(b) Any 2π-periodic sol'n is attracting

Let $x(t) = \phi(t, x_0)$

$z(t) = \frac{\partial}{\partial x}\phi(t, x_0)$

then z satisfies

$$\dot{z} = \frac{\partial}{\partial x} f(t, \phi(t, x_0)) z, \quad z(0) = 1$$

(where $f(t,x) = \cos t + x \sin t - x^3$)

$$\Rightarrow z(t) = \exp\left(\int_0^t \frac{\partial}{\partial x} f(s, \phi(s, x_0))\, ds\right)$$

$$= \exp\left(\int_0^t \sin(s) - 3x^2\, ds\right)$$

$$= \exp\left(-\cos(s)\Big|_0^t + \int_0^t -3x^2\, ds\right)$$

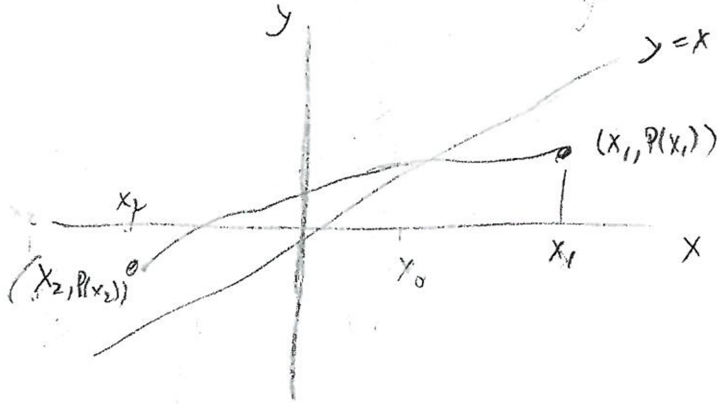
$P'(x_0) = \frac{\partial}{\partial x}\phi(2\pi, x_0) = z(2\pi)$

$\Rightarrow P'(x_0) = \exp\left(\int_0^{2\pi} -3x^2 ds\right) < 1$ *

$\Rightarrow$ $\phi(t, x_0)$ is an attracting solution $\Longrightarrow$

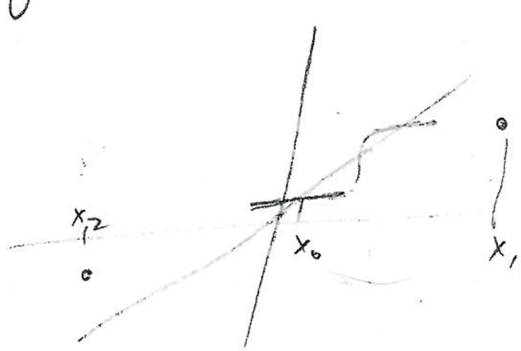
* unless $X \equiv 0$. However, $X \equiv 0$ is not a solution.

(c) From part (a), we know that there is a 2π-periodic solution.



from part (b), we know that any 2π-periodic solution is attracting, $P'(x_0) < 1$ for all 2π-periodic sol'ns $\phi(t, x_0)$

There can only be one 2π-periodic sol'n, because in order for another to exist, the graph of $y = P(x)$ must intersect $y = x$ again. In order for this to happen, $P'(x)$ must be greater than (or equal) 1 at some fixed pt.
(contradicts part b)



2. Define $A : C^0([a,b], \mathbb{R}) \to C^0([a,b], \mathbb{R})$ by

$$A\phi(t) = \int_a^t \phi(s)\,ds$$

A is a linear map

(a) Show A is bounded

$$\|A\phi(t)\| = \left\|\int_a^t \phi(s)\,ds\right\| \le |t-a| \sup_{a\le s\le t} \|\phi(s)\|$$

$$\le |b-a|\,\|\phi\|$$

this is true for all $t \in [a,b]$

$$\Rightarrow \quad \|A\phi\| \le |b-a|\,\|\phi\|$$

(b) Find $\|A\|$ and justify

from (a), we know $\|A\| \le |b-a|$

$$\|A\| = \sup_{\phi} \frac{\|A\phi\|}{\|\phi\|} \le |b-a|$$

Claim: $\|A\| = |b-a|$

I will show there is a $\phi \in C^0([a,b], \mathbb{R})$ for which $\dfrac{\|A\phi\|}{\|\phi\|} = |b-a|$

let $\phi(t) \equiv 1$, then $A\phi(t) = \int_a^t ds = t-a$

$$\|A\phi\| = \sup_{t\in[a,b]} \|A\phi(t)\| = |b-a|$$

3. Use Center Manifold Reduction to draw phase portrait

$$\dot{x} = y - x^2$$
$$\dot{y} = -2y + 2x^2 - 2xy$$

Linearization at $(0,0)$ gives

$$\lambda_1 = 0, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_2 = -2, \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

c.m. : $y = g(x) = Ax^2 + Bx^3 + \ldots$

$$\dot{y} = -2g(x) + 2x^2 - 2x\,g(x) = g'(x)\dot{x} = g'(x)\left(g(x) - x^2\right)$$

$$-2(Ax^2 + Bx^3 + \ldots) + 2x^2 - 2x(Ax^2 + Bx^3 + \ldots) = (2Ax + 3Bx^2 + \ldots)(Ax^2 + Bx^3 \ldots - x$$

x : ——

x^2 : $-2Ax^2 + 2x^2 = 0 \Rightarrow A = 1$

x^3 : $-2Bx^3 - 2Ax^3 = 2Ax(Ax^2 - x^2)$

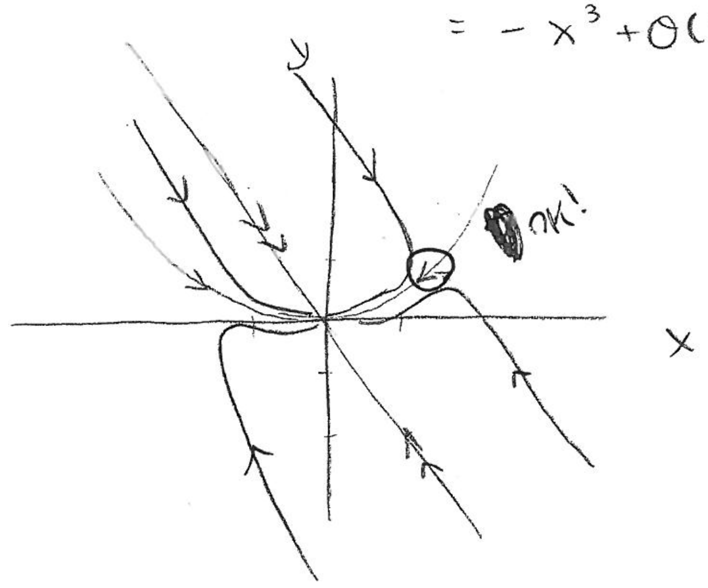
$$-2B - 2 = 2(0) = 0$$

$$B = -1$$

$$g(x) = x^2 - x^3 + O(x^4)$$

d.e. on c.m. : $\dot{x} = x^2 - x^3 + O(x^4) \cancel{- x^2}$
$= -x^3 + O(x^4)$

for $x < 0$, $\dot{x} > 0$
$x > 0$, $\dot{x} < 0$



4. Let B be an invertible $n \times n$ matrix, $G: \mathbb{R}^n \to \mathbb{R}^n$ C^1, $G(0) = 0$ and $DG(0) = 0$

Want to prove if $\|y\|$ small, $Bx + G(x) = y$ has unique small solution x

(a) First choose $\varepsilon > 0$ small enough s.t.

$\|x\| < \varepsilon \Rightarrow \|DG(x)\| \leq \frac{1}{2\|B^{-1}\|}$

We can do this because G is C^1, i.e. $DG(x)$ is continuous, and $DG(0) = 0$

(given $\underset{\hookrightarrow \frac{1}{2\|B^{-1}\|}}{\varepsilon_0} > 0$ $\exists$ $\underset{\varepsilon}{\overset{\shortparallel}{\delta}} > 0$ s.t. $\|x - 0\| < \delta \Rightarrow \|DG(x) - DG(0)\| < \varepsilon$

(b) Next let $\delta = \frac{\varepsilon}{2\|B^{-1}\|}$

If $\|y\| < \delta$, then T maps $\{x : \|x\| \leq \varepsilon\}$ into itself

$T(x,y) = B^{-1}(y - G(x))$

$\|T(x,y)\| = \|B^{-1}(y - G(x))\| \leq \|B^{-1}\| \left[\|y\| + \|G(x) - \overset{\overset{0}{\downarrow}}{G(0)}\|\right]$

(Corollary of) Mean Value Thm:

$\|G(x) - G(0)\| \leq \sup_{t \in [0,1]} \|DG(tx)\| \|x\|$

$\Rightarrow \|T(x,y)\| \leq \|B^{-1}\| \left[\frac{\varepsilon}{2\|B^{-1}\|} + \|DG(x)\| \|x\|\right]$

$\leq \|B^{-1}\| \left[\frac{\varepsilon}{2\|B^{-1}\|} + \frac{1}{2\|B^{-1}\|}\varepsilon\right] = \varepsilon$

$\|y\| < \delta$ and $\|x\| \leq \varepsilon \Rightarrow \|T(x,y)\| \leq \varepsilon$

$\Rightarrow T$ maps $\{x : \|x\| \leq \varepsilon\}$ into itself

$\Longrightarrow$

(c) Show that for each y with $\|y\|<\delta$, T is a contraction of $\{x : \|x\| \le \varepsilon\}$

let $x, x' \in \{x : \|x\| \le \varepsilon\}$

$$\|T(x,y)-T(x',y)\| = \|B^{-1}(G(x')-G(x))\|$$

Corollary of Mean Value Thm:

let $z = x'-x$

$$\|G(x+z)-G(x)\| \le \sup_{t\in[0,1]} \|DG(x+tz)\|\,\|z\|$$

$$\|T(x,y)-T(x',y)\| \le \|B^{-1}\| \sup_{0\le t\le 1} \|DG(x+t(x'-x))\|\,\|x'-x\|$$

$$\le \|B^{-1}\| \frac{1}{2\|B^{-1}\|}\|x'-x\|$$

$$\le \frac{1}{2}\|x-x'\|$$

since $\frac{1}{2} \in (0,1)$, $T(x,y)$ with $\|y\|<\delta$ is a contraction of $\{x : \|x\| \le \varepsilon\}$

(d) Let $H(y)$ be the fixed pt. of $T(\cdot,y)$

By the Contraction Mapping Thm. with parameters there is a <u>unique</u> fixed pt of $T(\cdot,y)$ s.t. $T(x,y)=x$, also $T(x,y)=x \Leftrightarrow Bx+G(x)=y$

$$\Rightarrow B(H(y))+G(H(y))=y$$

and also from the Contr. Map. Thm w/ Params since $T(\cdot,y)$ is C' (composition of C' fns)

$$\Rightarrow H(y) \text{ is } C'$$

Note that we've shown for each y with $\|y\|<\delta$ $\exists!$ x with $\|x\|<\varepsilon$ s.t. $Bx+G(x)=y$.