PROOF OF THE STABLE MANIFOLD THEOREM

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Consider the system \( \dot{x} = Ax + g(x) \), where \( x \in \mathbb{R}^n \), \( A \) is a hyperbolic \( n \times n \) matrix, \( g \) is \( C^1 \), \( g(0) = 0 \), and \( Dg(0) = 0 \).

From the definition of hyperbolic, there are complementary subspaces \( E^s \) and \( E^u \) of \( \mathbb{R}^n \), both invariant under \( A \), such that \( A|E^s \) has only eigenvalues with negative real part, and \( A|E^u \) has only eigenvalues with positive real part. There are corresponding linear projections \( \Pi_s : \mathbb{R}^n \to E^s \), \( \Pi_u : \mathbb{R}^n \to E^s \), such that any \( x \) equals \( \Pi_s x + \Pi_u x \). (This definition implies many things, such as \( \Pi_s x = x \) if and only if \( x \in E^s \), \( \Pi_u x = x \) if and only if \( x \in E^u \), \( \Pi_s x = 0 \) if and only if \( x \in E^u \), \( \Pi_u x = 0 \) if and only if \( x \in E^s \).)

\( \Pi_s \) and \( \Pi_u \) commute with \( A \). This is just another way of saying that \( E^s \) and \( E^u \) are invariant under \( A \):

\[
\Pi_s Ax = A\Pi_s x, \quad \Pi_u Ax = A\Pi_u x.
\]

\( E^s \) and \( E^u \) are both invariant under \( e^{tA} \), so \( \Pi_s \) and \( \Pi_u \) commute with \( A \):

\[
\Pi_s e^{tA} x = e^{tA}\Pi_s x, \quad \Pi_u e^{tA} x = e^{tA}\Pi_u x.
\]

Recall that there exist constants \( K \geq 1 \) and \( \alpha > 0 \) such that for all \( x \in \mathbb{R}^n \),

\[
|e^{tA}\Pi_s x| \leq Ke^{-\alpha t} |\Pi_s x| \text{ for all } t \geq 0, \quad |e^{-tA}\Pi_u x| \leq Ke^{-\alpha t} |\Pi_u x| \text{ for all } t \geq 0.
\]

Let \( U_\epsilon = \{ x \in \mathbb{R}^n : |x| \leq \epsilon \}, \ V_\delta = \{ x \in \mathbb{R}^n : |\Pi_s x| < \delta \} \).

Recall that if \( 0 \) is an equilibrium of \( \dot{x} = f(x) \) and \( U \) is a neighborhood of \( 0 \), then the local stable set of \( 0 \) with respect to \( U \) is \( W_{loc}^s(0) = \{ x \in U : \phi(t, x) \in U \text{ for all } t \geq 0 \text{ and } \phi(t, x) \to 0 \text{ as } t \to \infty \} \).

**Theorem 1** (Stable Manifold Theorem). Consider \( \dot{x} = Ax + g(x) \) as above. For sufficiently small \( \epsilon > 0 \) there exists \( \delta, 0 < \delta < \epsilon \), such that the local stable set of \( 0 \) with respect to \( U_\epsilon \), intersected with \( V_\delta \), is the graph of a \( C^1 \) function \( k \) from \( \{ x_s \in E^s : |x_s| < \delta \} \) to \( E^u \). Moreover, \( k(0) = 0 \) and \( Dk(0) = 0 \).

The theorem says essentially that \( W_{loc}^s(0) \) is a \( C^1 \) manifold, with dimension equal to that of \( E^s \), that is tangent to \( E^s \) at \( 0 \).

More generally, if \( g \) is \( C^r \) (respectively \( C^\infty \), respectively analytic), then \( k \) is \( C^r \) (respectively \( C^\infty \), respectively analytic).

The proof is an application of the Contraction Mapping Theorem with Parameters. But first we need a lemma.

We shall use \( \mathbb{R}_+ \) to mean the interval \([0, \infty)\).

**Lemma 2.** Consider the inhomogeneous linear differential equation

\[
\dot{x} = Ax + h(t) \quad (1)
\]

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with $A$ as above and $h : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ a bounded continuous function. Let $x_s \in E^s$. Then there is a unique function $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ such that (i) $x$ is a solution of (1), (ii) $x$ is bounded, and (iii) $\Pi_s x(0) = x_s$. Moreover,

$$x(t) = e^{tA}x_s + \int_0^t e^{(t-s)A}\Pi_s h(s) \, ds + \int_t^{\infty} e^{(t-s)A}\Pi_u h(s) \, ds. \tag{2}$$

The first two terms on the right give the stable part of $x(t)$. The last term gives the unstable part.

**Proof.** Let $\tau \in \mathbb{R}$. Any solution of (1) can be written

$$x(t) = e^{(t-\tau)A}x(\tau) + \int_\tau^t e^{(t-s)A}h(s) \, ds. \tag{3}$$

Take $\tau = 0$ in (3) and apply $\Pi_s$ to both sides. We get

$$\Pi_s x(t) = e^{tA}\Pi_s x(0) + \int_0^t e^{(t-s)A}\Pi_s h(s) \, ds.$$

In order to satisfy (iii), we have

$$\Pi_s x(t) = e^{tA}x_s + \int_0^t e^{(t-s)A}\Pi_s h(s) \, ds. \tag{4}$$

Now apply $\Pi_u$ to both sides of (3). We get

$$\Pi_u x(t) = e^{(t-\tau)A}\Pi_u x(\tau) + \int_\tau^t e^{(t-s)A}\Pi_u h(s) \, ds. \tag{5}$$

Think of $t$ as fixed and let $\tau \to \infty$. If $x$ is bounded, then in the first summand, $t - \tau \to -\infty$ and $\Pi_u x(\tau)$ stays bounded. Therefore the first summand approaches 0, so we obtain

$$\Pi_u x(t) = \int_\infty^t e^{(t-s)A}\Pi_u h(s) \, ds. \tag{6}$$

Adding (4) and (6) we obtain (2).

What we have shown so far is uniqueness: if $x$ satisfies (i), (ii), and (iii), then $x(t)$ is given by (3). To show existence, just check that (3) does in fact satisfy (i), (ii), and (iii). (i) and (iii) are pretty clear, we’ll check (ii) shortly. □

To prove the Stable Manifold Theorem, define $T : C^0(\mathbb{R}_+, \mathbb{R}^n) \rightarrow C^0(\mathbb{R}_+, \mathbb{R}^n)$ by

$$Th(t) = \int_0^t e^{(t-s)A}\Pi_s h(s) \, ds + \int_t^{\infty} e^{(t-s)A}\Pi_u h(s) \, ds.$$

$Th(t)$ is the last two terms in (2). Actually, it’s not yet clear that $Th$ is a bounded function, but we will show that shortly.

**Lemma 3.** $T$ is a bounded linear map with $\|T\| \leq \frac{2K}{\alpha}$.
Proof. $T$ is clearly linear. To show that $T$ is bounded, calculate (assuming $\Pi_s = \Pi_u = 1$ for simplicity)

$$|Th(t)| \leq \int_0^t Ke^{-\alpha(t-s)}|h| \, ds + \int_t^\infty Ke^{-\alpha(s-t)}|h| \, ds$$

$$= K|h| \left( e^{-\alpha t} \cdot \left[ e^{\alpha s} \left[ e^{-\alpha t} \cdot \left( e^{\alpha s} - 1 \right) + e^{\alpha s} \right] \right] \right) = K|h| \left( e^{-\alpha t} \left( e^{\alpha t} - \frac{1}{\alpha} \right) + e^{\alpha t} \right) \leq \frac{2K}{\alpha} |h|.$$ 

Therefore $|Th| \leq \frac{2K}{\alpha} |h|.$ This shows both that $Th$ is a bounded function, and that $T$ is a bounded linear map with $\|T\| \leq \frac{2K}{\alpha}$. 

Now we prove the Stable Manifold Theorem. Suppose $x: \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is a bounded solution of $\dot{x} = Ax + g(x),$ i.e., $\dot{x} = Ax + g(x(t)).$ Since $x(t)$ is bounded, so is $g(x(t)).$ Hence, if $\Pi_s x(0) = x_s,$ Lemma 2 tells us that

$$x(t) = e^{tA}x_s + \int_0^t e^{(t-s)A}\Pi_s g(x(s)) \, ds + \int_t^\infty e^{(t-s)A}\Pi_u g(x(s)) \, ds.$$ 

(7)

To express (7) more compactly, define $N: C^0(\mathbb{R}_+, \mathbb{R}^n) \rightarrow C^0(\mathbb{R}_+, \mathbb{R}^n)$ by $N(x)(t) = g(x(t)).$ Then define $F: E^s \times C^0(\mathbb{R}_+, \mathbb{R}^n) \rightarrow C^0(\mathbb{R}_+, \mathbb{R}^n)$ by

$$F(x_s, x)(t) = e^{At}x_s + T \circ N(x)(t).$$ 

Equation (7) can now be written

$$x = F(x_s, x).$$ 

(8)

In other words, for given $x_s,$ a solution of $\dot{x} = Ax + g(x)$ that is bounded on $\mathbb{R}_+$ and has $\Pi_s x(0) = x_s$ is a fixed point of the mapping $F(x_s, \cdot)$ from $C^0(\mathbb{R}_+, \mathbb{R}^n)$ to itself. This is the situation of the Contraction Mapping Theorem with Parameters.

Now $T$ is bounded linear and $N$ is $C^1,$ so $T \circ N$ is $C^1.$ The map $x_s \rightarrow e^{At}x_s$ from $E^s$ to $C^0(\mathbb{R}_+, \mathbb{R}^n)$ is bounded linear. Therefore $F$ is $C^1.$

Choose $\epsilon > r$ small enough so that

$$\sup_{|x| \leq \epsilon} \|Dg(x)\| \leq \frac{\alpha}{4K}.$$ 

(9)

In this expression $x$ is just a point in $\mathbb{R}^n.$ We can do this because $Dg(0) = 0,$ and $Dg(x)$ depends continuously on $x$ because $g$ is $C^1.$

Let $\delta = \frac{\epsilon}{4K}.$ Since $K \geq 1,$ we have $0 < \delta < \epsilon.$

Let

$$B = \{ x_s \in E^s : |x_s| < \delta \}, \quad W = \{ x \in C^0(\mathbb{R}_+, \mathbb{R}^n) : |x| \leq \epsilon \}.$$ 

We claim that for each $x_s \in B,$ $F(x_s, \cdot)$ (i) maps $W$ into itself, and (ii) is a contraction of $W$ with contraction constant $\frac{1}{2}.$

To show (i), let $x_s \in B$ and $x \in W.$ Then

$$|e^{At}x_s| \leq K e^{-\alpha t}|x_s| \leq K|x_s|.$$ 

Also, if $x \in \mathbb{R}^n$ and $|x| \leq \epsilon,$ then

$$|g(x)| = |g(x) - 0| = |g(x) - g(0)| \leq \sup_{0 \leq s \leq 1} \|Dg(sx)\| |x| \leq \frac{\alpha}{4K} \epsilon.$$ 

Therefore if $x \in W$ then $|N(x)| \leq \frac{\alpha}{4K} \epsilon.$
We conclude that if \( x_s \in B \) and \( x \in W \),

\[
|F(x, s, x)| \leq |e^{At}x_s| + |TN(x)| \leq K|x_s| + \|T\| |N(x)| \leq K \frac{\epsilon}{2K} + 2K \frac{\alpha}{4K} \epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

This proves (i).

To show (ii), let \( x_s \in B \) and \( x, y \in W \). Then

\[
|g(x(t)) - g(y(t))| \leq \sup_{0 \leq s \leq 1} \left\| Dg \left( x(t) + s(y(t) - x(t)) \right) \right\| |x(t) - y(t)| \leq \frac{\alpha}{4K} |x - y|.
\]

Therefore \( |N(x) - N(y)| \leq \frac{\alpha}{4K} |x - y| \), so

\[
|F(x, x) - F(x, y)| = |TN(x) - TN(y)| = |T(N(x) - N(y))| \leq \|T\| |N(x) - N(y)| \leq \frac{2K \alpha}{4K} |x - y| \leq \frac{1}{2} |x - y|.
\]

This proves (ii).

For each \( x_s \in B \) let \( \psi(x_s) \in C^0(\mathbb{R}_+, \mathbb{R}^n) \) denote the fixed point of \( F(x_s, \cdot) \) in \( W \). The Contraction Mapping Theorem with Parameters says that the mapping

\[
\psi : B \to W \subset C^0(\mathbb{R}_+, \mathbb{R}^n)
\]

is \( C^1 \).

By construction \( \Pi_s \psi(x_s)(0) = x_s \). We are interested in \( \Pi_u \psi(x_s)(0) \). Define \( k : E^s \to E^u \) by

\[
k(x_s) = \Pi_u \psi(x_s)(0) = \Pi_u \circ ev_0 \circ \psi(x_s),
\]

where \( ev_0 : C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^n \) is the bounded linear map that takes the function \( h(t) \) to \( h(0) \). \( ev_0 \) means “evaluate at \( t = 0 \)”.

Now a fixed point of \( F(0, \cdot) \) is certainly \( x \equiv 0 \) (it satisfies all the conditions). Since \( x \equiv 0 \) is in \( W \), we must have \( \psi(0) = 0 \). (The first 0 is a point in \( E^s \), the second is a constant function.) Therefore

\[
k(0) = \Pi_u \circ ev_0 \circ \psi(0) = 0.
\]

Now

\[
Dk(0) = \Pi_u \circ ev_0 \circ D\psi(0).
\]

To compute \( D\psi(0) \), note that \( \psi(x_s) = F(x_s, \psi(x_s)) \) so

\[
D\psi(x_s) = D_1F(x_s, \psi(x_s)) + D_2F(x_s, \psi(x_s))D\psi(x_s).
\]

Taking \( x_s = 0 \) and applying both sides to a vector \( h \in E^s \), we obtain

\[
D\psi(0)h = D_1F(0, 0)h + D_2F(0, 0)D\psi(0)h = e^{At}h + T \circ DN(0)h = e^{At}h
\]

because \( DN(0) = 0 \) (a consequence of \( Dg(0) = 0 \)). Therefore

\[
Dk(0)h = \Pi_u \circ ev_0 \circ D\psi(0)h = \Pi_u \circ ev_0(e^{At}h) = \Pi_u h = 0.
\]

because \( h \in E^s \). Therefore \( Dk(0) = 0 \).

This completes the proof of the Stable Manifold Theorem.