The gradient of $G : M \to \mathbb{R}$ with respect to the Riemannian metric g is the vector field, denoted by $\operatorname{grad} G$, such that

$$dG_p(V) = g_p(V, \operatorname{grad} G) \tag{1.28}$$

for each point $p \in M$ and every tangent vector $V \in T_pM$. The associated gradient system on the manifold is the differential equation $\dot{p} = \operatorname{grad} G(p)$.
(a) Prove that the gradient vector field is uniquely defined.
(b) Prove that if the Riemannian metric g on $\mathbb{R}^3$ is the usual inner product at each point of $\mathbb{R}^3$, then the invariant definition (1.28) of gradient agrees with the Euclidean gradient.

Consider the upper half-plane of $\mathbb{R}^2$ with the Riemannian metric

$$g_{(x,y)}(V, W) = y^{-2}\langle V, W\rangle \tag{1.29}$$

where the angle brackets denote the usual inner product. The upper half-plane with the metric g is called the Poincaré or Lobachevsky plane; its geodesics are vertical lines and arcs of circles whose centers are on the x-axis. The geometry is non-Euclidean; for example, if p is a point not on such a circle, then there are infinitely many such circles passing through p that are parallel to (do not intersect) the given circle (see Exercise 3.11).
(c) Determine the gradient of the function $G(x, y) = x^2 + y^2$ with respect to the Riemannian metric (1.29) and draw the phase portrait of the corresponding gradient system on the upper half-plane. Also, compare this phase portrait with the phase portrait of the gradient system with respect to the usual metric on the plane.

If S is a submanifold of $\mathbb{R}^n$, then S inherits a Riemannian metric from the usual inner product on $\mathbb{R}^n$.
(d) Suppose that $F : \mathbb{R}^n \to \mathbb{R}$. What is the relationship between the gradient of F on $\mathbb{R}^n$ and the gradient of the function F restricted to S with respect to the inherited Riemannian metric (see Exercise 1.116)?

Hamiltonian systems on manifolds are defined in essentially the same way as gradient systems except that the Riemannian metric is replaced by a symplectic form. Although these objects are best described and analyzed using the calculus of differential forms (see [12], [89], and [213]), they are easy to define. Indeed, a *symplectic form* on a manifold is a smooth assignment of a bilinear, skew-symmetric, nondegenerate 2-form in each tangent space. A 2-form ω on a vector space X is nondegenerate provided that $y = 0$ is the only element of X such that $\omega(x, y) = 0$ for all $x \in X$. Prove: If a manifold has a symplectic form, then the dimension of the manifold is even.

Suppose that M is a manifold and ω is a symplectic form on M. The Hamiltonian vector field associated with a smooth scalar function H defined on M is the unique vector field X_H such that, for every point $p \in M$ and all tangent vectors V at p, the following identity holds:

$$dH_p(V) = \omega_p(X_H, V). \tag{1.30}$$

(e) Let $M := \mathbb{R}^{2n}$, view $\mathbb{R}^{2n}$ as $\mathbb{R}^n \times \mathbb{R}^n$ so that each tangent vector V on M is decomposed as $V = (V_1, V_2)$ with $V_1, V_2 \in \mathbb{R}^n$, and define

$$\omega(V, W) := (V_1, V_2)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}.$$

Show that ω is a symplectic form on M and Hamilton's equations are produced by the invariant definition (1.30) of the Hamiltonian vector field.
(f) Push forward the Euclidean gradient (1.27) of the function $G : \mathbb{R}^3 \to \mathbb{R}$ to the image of a cylindrical coordinate map, define

$$\mathcal{G}(r, \theta, z) = G(r\cos\theta, r\sin\theta, z),$$

and show that the push forward gives the result

$$\operatorname{grad} \mathcal{G} = \Big(\frac{\partial \mathcal{G}}{\partial r}, \frac{1}{r^2}\frac{\partial \mathcal{G}}{\partial \theta}, \frac{\partial \mathcal{G}}{\partial z}\Big). \tag{1.31}$$

(In practice, the function $\mathcal{G}$ is usually again called G. These two functions are local representations of the same function in two different coordinate systems.)
(g) Recall the formula for the gradient in cylindrical coordinates from vector analysis; namely,

$$\operatorname{grad} \mathcal{G} = \frac{\partial \mathcal{G}}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \mathcal{G}}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \mathcal{G}}{\partial z}\mathbf{e}_z. \tag{1.32}$$

Show that the gradient vector fields (1.31) and (1.32) coincide.
(h) Express the usual inner product in cylindrical coordinates, and use the invariant definition of the gradient to determine the gradient in cylindrical coordinates.
(i) Repeat part (h) for spherical coordinates.

Exercise 1.140. [Electrostatic Potential] Suppose that two point charges with opposite signs, each with charge q, placed a units apart and located symmetrically with respect to the origin on the z-axis in space, produce the electrostatic potential

$$G_0(x, y, z) = kq\big[(x^2 + y^2 + (z - \tfrac{a}{2})^2)^{-1/2} - (x^2 + y^2 + (z + \tfrac{a}{2})^2)^{-1/2}\big]$$

where $k > 0$ is a constant and $q > 0$. If we are interested only in the field far from the charges, the "far field," then a is relatively small and therefore the first nonzero term of the Taylor series of the electrostatic potential with respect to a at $a = 0$ gives a useful approximation of G_0. This approximation, an example of a "far field approximation," is called the *dipole potential* in Physics (see [87, Vol. II, 6-1]). Show that the dipole potential is given by

$$G(x, y, z) = kqaz(x^2 + y^2 + z^2)^{-3/2}.$$

By definition, the electric field E produced by the dipole potential associated with the two charges is $E := -\operatorname{grad} G$. Draw the phase portrait of the differential equation $\dot{u} = E(u)$ whose orbits are the "dipole" lines of force. Discuss the stability of all rest points. Hint: Choose a useful coordinate system that reduces the problem to two dimensions.

Blow Up at a Rest Point

As an application of polar coordinates, let us determine the phase portrait of the differential equation in the Cartesian plane given by

$$\dot{x} = x^2 - 2xy, \qquad \dot{y} = y^2 - 2xy, \tag{1.33}$$

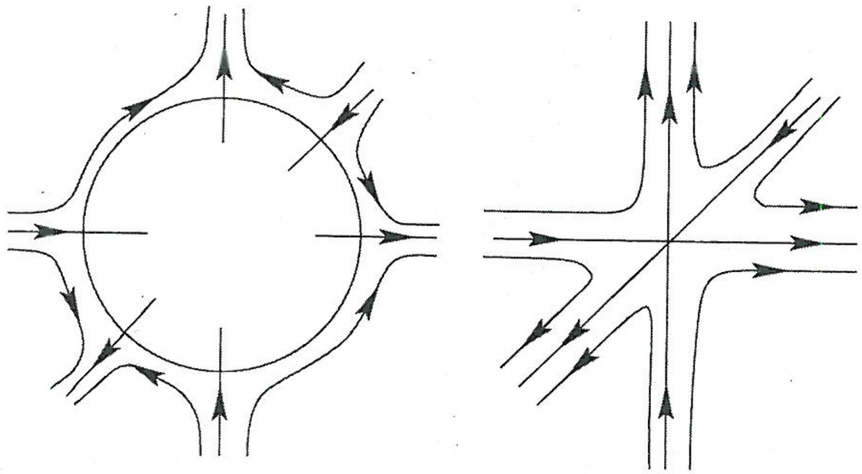

Figure 1.20: Phase portrait for the differential equation (1.34) on the upper half of the phase cylinder and its "blowdown" to the Cartesian plane.

(see [76]). This system has a unique rest point at the origin that is not hyperbolic. In fact, the system matrix for the linearization at the origin vanishes. Thus, linearization provides no information about the phase portrait of the system near the origin.

Because the polar coordinate representation of a plane vector field is always singular at the origin, we might expect that the polar coordinate representation of a planar vector field is not particularly useful to determine the phase portrait near the origin. But this is not the case. Often polar coordinates are the best way to analyze the vector field near the origin. The reason is that the desingularized vector field in polar coordinates is a smooth extension to the singular line represented as the equator of the phase cylinder. All points on the equator are collapsed to the single rest point at the origin in the Cartesian plane. Or, as we say, the equator is the *blowup* of the rest point. This extension is valuable because the phase portrait of the vector field near the original rest point corresponds to the phase portrait on the phase cylinder near the equatorial circle. Polar coordinates and desingularization provide a mathematical microscope for viewing the local behavior near the "Cartesian" rest point.

The desingularized polar coordinate representation of system (1.33) is

$$\begin{aligned} \dot{r} &= r^2(\cos^3\theta - 2\cos^2\theta\sin\theta - 2\cos\theta\sin^2\theta + \sin^3\theta), \\ \dot{\theta} &= 3r(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta). \end{aligned} \tag{1.34}$$

For this particular example, both components of the vector field have r as a common factor. From our discussion of reparametrization, we know that the system with this factor removed has the same phase portrait as the original differential equation in the portion of the phase cylinder where $r > 0$. Of course, when we "blow down" to the Cartesian plane, the push forward of the reparametrized vector field has the same phase portrait as the original vector field in the punctured plane; exactly the set where the original phase portrait is to be constructed.

Let us note that after division by r, the differential equation (1.34) has several *isolated* rest point on the equator of the phase cylinder. In fact, because this differential equation restricted to the equator is given by

$$\dot{\theta} = 3\cos\theta\sin\theta(\sin\theta - \cos\theta),$$

we see that it has six rest points with the following angular coordinates:

$$0, \quad \frac{\pi}{4}, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{5\pi}{4}, \quad \frac{3\pi}{2}.$$

The corresponding rest points for the reparametrized system are all hyperbolic. For example, the system matrix at the rest point $(r,\theta) = (0, \frac{\pi}{4})$ is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

It has the negative eigenvalue $-1/\sqrt{2}$ in the positive direction of the Cartesian variable r on the cylinder and the positive eigenvalue $3/\sqrt{2}$ in the positive direction of the angular variable. This rest point is a hyperbolic saddle. If each rest point on the equator is linearized in turn, the phase portrait on the cylinder and the corresponding blowdown of the phase portrait on the Cartesian plane are found to be as depicted in Figure 1.20. Hartman's theorem can be used to construct a proof of this fact.

The analysis of differential equation (1.33) is very instructive, but perhaps somewhat misleading. Often, unlike this example, the blowup procedure produces a vector field on the phase cylinder where some or all of the rest points are not hyperbolic. Of course, in these cases, we can treat the polar coordinates near one of the nonhyperbolic rest points as Cartesian coordinates; we can translate the rest point to the origin; and we can blow up again. If, after a finite number of such blowups, all rest points of the resulting vector field are hyperbolic, then the local phase portrait of the original vector field at the original nonhyperbolic rest point can be determined. For masterful treatments of this subject and much more, see [19], [75], [76], and [219].

The idea of blowup and desingularization are far-reaching ideas in mathematics. For example, these ideas seem to have originated in algebraic geometry, where they play a fundamental role in understanding the structure of algebraic varieties [29].

Compactification at Infinity

The orbits of a differential equation on $\mathbb{R}^n$ may be unbounded. One way to obtain some information about the behavior of such solutions is to (try to) *compactify* the Cartesian space, so that the vector field is extended to a new manifold that contains the "points at infinity." This idea, due to Henri Poincaré [185], has been most successful in the study of planar systems given by polynomial vector fields, also called polynomial systems (see [7, p. 219] and [99]). In this section we will give a brief description of the compactification process for such planar systems. We will again use the manifold concept and the idea of reparametrization.

Let us consider a plane vector field, which we will write in the form

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y). \tag{1.35}$$

To study its phase portrait "near" infinity, let us consider the unit sphere $\mathbb{S}^2$; that is, the two-dimensional submanifold of $\mathbb{R}^3$ defined by

$$\mathbb{S}^2 := \{(x, y, z) : x^2 + y^2 + z^2 = 1\},$$

and the tangent plane Π at its north pole; that is, the point with coordinates $(0, 0, 1)$. The push forward of system (1.35) to Π by the natural map $(x, y) \mapsto (x, y, 1)$ is

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y), \qquad \dot{z} = 0. \tag{1.36}$$

The idea is to "project" differential equation (1.36) to the unit sphere by central projection; then the behavior of the system near infinity is the same as the behavior of the projected system near the equator of the sphere.

Central projection is defined as follows: A point $p \in \Pi$ is mapped to the sphere by assigning the unique point on the sphere that lies on the line segment from the origin in $\mathbb{R}^3$ to the point p. To avoid a vector field specified by *three* components, we will study the projected vector field restricted to a coordinate system on the sphere where the vector field is again planar. Also, to obtain the desired compactification, we will choose local coordinates defined in open sets that contain portions of the equator of the sphere.

The central projection map $Q : \Pi \to \mathbb{S}^2$ is given by

$$Q(x, y, 1) = (x(x^2 + y^2 + 1)^{-1/2}, y(x^2 + y^2 + 1)^{-1/2}, (x^2 + y^2 + 1)^{-1/2}).$$

One possibility for an appropriate coordinate system on the Poincaré sphere is a spherical coordinate system; that is, one of the coordinate charts that is compatible with the map

$$(\rho, \phi, \theta) \mapsto (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \tag{1.37}$$

(see display (1.24)). For example, if we restrict to the portion of the sphere where $x > 0$, then one such coordinate map is given by

$$\Psi(x, y, z) := (\arccos(z), \arctan\left(\frac{y}{x}\right)).$$

The transformed vector field on the sphere is the push forward of the vector field X that defines the differential equation on Π by the map $\Psi \circ Q$. In view of equation (1.37) and the restriction to the sphere, the inverse of this composition is the transformation P given by

$$P(\phi, \theta) = \left(\frac{\sin\phi}{\cos\phi} \cos\theta, \ \frac{\sin\phi}{\cos\phi} \sin\theta\right).$$

Thus, the push forward of the vector field X is given by

$$DP(\phi, \theta)^{-1} X(P(\phi, \theta)).$$

Of course, we can also find the transformed vector field simply by differentiating with respect to t in the formulas

$$\phi = \arccos((x^2 + y^2 + 1)^{-1/2}), \qquad \theta = \arctan\left(\frac{y}{x}\right).$$

If the vector field is polynomial with maximal degree k, then after we evaluate the polynomials f and g in system (1.36) at $P(\phi, \theta)$ and take into account multiplication by the Jacobian matrix, the denominator of the resulting expressions will contain $\cos^{k-1}\phi$ as a factor. Note that $\phi = \frac{\pi}{2}$ corresponds to the equator of the sphere and $\cos(\frac{\pi}{2}) = 0$. Thus, the vector field in spherical coordinates is desingularized by a reparametrization of time that corresponds to multiplication of the vector field defining the system by $\cos^{k-1}\phi$. This desingularized system ([53])

$$\dot{\phi} = (\cos^{k+1}\phi)(\cos\theta\, f + \sin\theta\, g), \qquad \dot{\theta} = \frac{\cos^k \phi}{\sin\phi}(\cos\theta\, g - \sin\theta\, f) \tag{1.38}$$

is smooth at the equator of the sphere, and it has the same phase portrait as the original centrally projected system in the upper hemisphere. Therefore, we can often determine the phase portrait of the original vector field "at infinity" by determining the phase portrait of the desingularized vector field on the equator. Note that because the vector field corresponding to system (1.38) is everywhere tangent to the equator, the equator is an invariant set for the desingularized system.

Spherical coordinates are global in the sense that all the spherical coordinate systems have coordinate maps that are local inverses for the fixed spherical wrapping function (1.37). Thus, the push forward of the original vector field will produce system (1.38) in every spherical coordinate system.

There are other coordinate systems on the sphere that have also proved useful for the compactification of plane vector fields. For example, the right hemisphere of $\mathbb{S}^2$; that is, the subset $\{(x, y, z) : y > 0\}$ is mapped diffeomorphically to the plane by the coordinate function defined by

$$\Psi_1(x, y, z) = \Big(\frac{x}{y}, \frac{z}{y}\Big).$$

Also, the map $\Psi_1 \circ Q$, giving the central projection in these coordinates, is given by

$$(x, y, 1) \mapsto \Big(\frac{x}{y}, \frac{1}{y}\Big).$$

Thus, the local representation of the central projection in this chart is obtained using the coordinate transformations

$$u = \frac{x}{y}, \qquad v = \frac{1}{y}.$$

Moreover, a polynomial vector field of degree k in these coordinates can again be desingularized at the equator by a reparametrization corresponding to multiplication of the vector field by v^{k-1}. In fact, the desingularized vector field has the form

$$\dot{u} = v^k\Big(f\big(\frac{u}{v}, \frac{1}{v}\big) - ug\big(\frac{u}{v}, \frac{1}{v}\big)\Big), \qquad \dot{v} = -v^{k+1}g\big(\frac{u}{v}, \frac{1}{v}\big).$$

The function Ψ_1 restricted to $y < 0$ produces the representation of the central projection in the left hemisphere. Similarly, the coordinate map

$$\Psi_2(x, y, z) = \Big(\frac{y}{x}, \frac{z}{x}\Big)$$

on the sphere can be used to cover the remaining points, near the equator in the upper hemisphere, with Cartesian coordinates (x, y, z) where $y = 0$ but $x \neq 0$.

The two pairs of charts just discussed produce two different local vector fields. Both of these are usually required to analyze the phase portrait near infinity. Also, it is very important to realize that if the degree k is even, then multiplication by v^{k-1} in the charts corresponding respectively to $x < 0$ and $y < 0$ *reverses* the original direction of time.

As an example of compactification, let us consider the phase portrait of the quadratic planar system given by

$$\dot{x} = 2 + x^2 + 4y^2, \qquad \dot{y} = 10xy. \tag{1.39}$$

This system has no rest points in the finite plane.

In the chart corresponding to $v > 0$ with the chart map Ψ_1, the desingularized system is given by

$$u' = 2v^2 - 9u^2 + 4, \qquad v' = -10uv \tag{1.40}$$

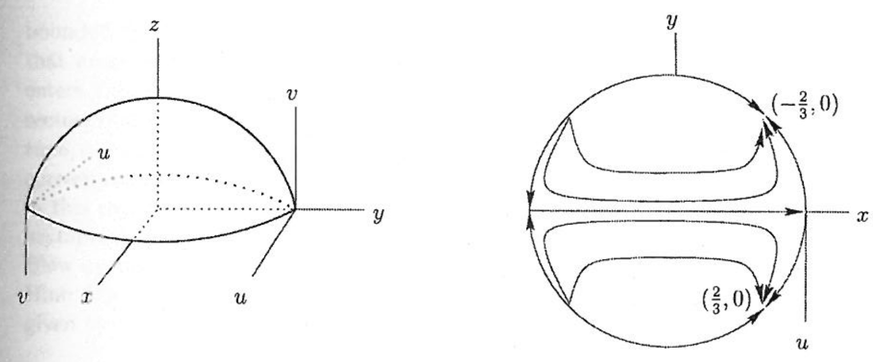

Figure 1.21: Phase portrait on the Poincaré sphere for the differential equation (1.39).

where the symbol "$'$" denotes differentiation with respect to the new independent variable after reparametrization. The first order system (1.40) has rest points with coordinates $(u, v) = (\pm\frac{2}{3}, 0)$. These rest points lie on the u-axis: the set in our chart that corresponds to the equator of the Poincaré sphere. Both rest points are hyperbolic. In fact, $(\frac{2}{3}, 0)$ is a hyperbolic sink and $(-\frac{2}{3}, 0)$ is a hyperbolic source.

In the chart with $v < 0$ and chart map Ψ_1, the reparametrized local system is given by the differential equation (1.40). But, because $k = 2$, the direction of "time" has been reversed. Thus, the sink at $(\frac{2}{3}, 0)$ in this chart corresponds to a source for the original vector field centrally projected to the Poincaré sphere. The rest point $(-\frac{2}{3}, 0)$ corresponds to a sink on the Poincaré sphere.

We have now considered all points on the Poincaré sphere except those on the great circle given by the equation $y = 0$. For these points, we must use the charts corresponding to the map Ψ_2. In fact, there is a hyperbolic saddle point at the origin of each of these coordinate charts, and these rest points correspond to points on the equator of the Poincaré sphere. Of course, the other two points already discussed are also rest points in these charts.

The phase portrait of the compactification of system (1.39) is shown in Figure 1.21. Because the x-axis is an invariant manifold for the original vector field, the two saddles at infinity are connected by a heteroclinic orbit.

Exercise 1.141. Prove that $\mathbb{S}^2$ is a two-dimensional submanifold of $\mathbb{R}^3$.

Exercise 1.142. Use spherical coordinates to determine the compactification of the differential equation (1.39) on the Poincaré sphere.