PERSISTENCE OF PERIODIC SOLUTIONS IN A ONE-PARAMETER FAMILY

S. SCHECTER

Consider

$$\dot{x} = f(x) + \mu g(x), \quad x \in \mathbb{R}^2, \quad \mu \text{ small}.$$

(More generally one could consider $\dot{x} = f(x)+\mu g(x,\mu)$ with only small changes. This simpler situation is often considered in order to simplify the notation.) Denote the flow by

$$x = \phi(t, y, \mu),$$

i.e., $x(t) = \phi(t, y, \mu)$ is the solution with $x(0) = y$ and parameter value μ.

Assume: $\dot{x} = f(x)$ has a one-parameter family of closed orbits. We want to know if any "persist" when $\mu \neq 0$.

Let y_0 be a point on one of the closed orbits, and let Σ be a line segment through y_0 that is perpendicular to the closed orbit of $\dot{x} = f(x)$ through y_0. Let

$$v_0 = \begin{pmatrix} -f_2(y_0) \\ f_1(y_0) \end{pmatrix} / \|f(y_0)\|.$$

The vector v_0 is a unit vector that is perpendicular to $f(y_0)$, so it is perpendicular at y_0 to the closed orbit of $\dot{x} = f(x)$ through y_0. Then

$$\Sigma = \{y(\xi) = y_0 + \xi v_0, \ \xi \text{ small}\},$$

i.e., ξ parameterizes Σ. Notice that

$$v_0 \cdot (y(\xi) - y_0) = v_0 \cdot \xi v_0 = \xi$$

because $v_0 \cdot v_0 = 1$.

The Poincaré map of $\dot{x} = f(x) + \mu g(x)$ on Σ, using the parameter ξ, is just

$$P(\xi, \mu) = v_0 \cdot \big(\phi(T(\xi, \mu), y(\xi), \mu) - y_0\big),$$

where $T(\xi, \mu)$ is the time it takes for the solution of $\dot{x} = f(x)+\mu g(x)$ that starts at $y(\xi) \in \Sigma$ to return to Σ. Since $P(\xi, \mu) = \xi$ (because the orbits of $\dot{x} = f(x)$ that pass near y_0 are closed), we can write

$$P(\xi, \mu) = \xi + \mu\Delta(\xi, \mu), \quad \Delta(\xi, 0) = \frac{\partial P}{\partial \mu}(\xi, 0).$$

Lemma 1. *If $\Delta(\xi_1, 0) = 0$ and $\frac{\partial \Delta}{\partial \xi}(\xi_1, 0) = 0$, then there is a curve $\xi(\mu)$, μ small, with $\xi(0) = \xi_1$, such that $P(\xi(\mu), \mu) = 0$.*

Proof. By the Implicit Function Theorem, there is a curve $\xi(\mu)$, μ small, with $\xi(0) = \xi_1$, such that $\Delta(\xi(\mu), \mu) = 0$. The result follows. $\square$

Thus the closed orbit of $\dot{x} = f(x)$ through $y(\xi_1)$ "persists," in the sense that for small μ, $\dot{x} = f(x) + \mu g(x)$ has a closed orbit through $y(\xi(\mu))$.

Date: April 7, 2013.

Lemma 2. *Let $a(\xi)$, $b(\xi)$, and $c(\xi)$ be C^1 functions with $c(\xi) \neq 0$ for all ξ and $a(\xi) = c(\xi)b(\xi)$. Then*

$$a(\xi_1) = 0 \text{ and } a'(\xi_1) \neq 0 \text{ if and only if } b(\xi_1) = 0 \text{ and } b'(\xi_1) \neq 0.$$

The proof is left to you.

Let $x(t, \xi, \mu)$ denote the solution of $\dot{x} = f(x) + \mu g(x)$ whose value at $t = 0$ is $y(\xi)$, i.e.,

$$x(t, \xi) = \phi(t, y(\xi), \mu).$$

(ξ and μ are parameters.) The linearization of $\dot{x} = f(x)$ along $x(t, \xi, 0)$ is, as usual,

$$\dot{X} = Df(x(t, \xi, 0))X.$$

Denote the propagator of $\dot{X} = Df(x(t, \xi, 0))X$ by $\Phi(t, s, \xi)$. (Again ξ is a parameter.) In other words, if $X(t)$ is a solution of $\dot{X} = Df(x(t, \xi, 0))X$, then for any t and s,

$$X(t) = \Phi(t, s, \xi)X(s).$$

Then the adjoint equation

$$\dot{w} = -wDf(x(t, \xi, 0))$$

has the same propagator:

$$w(s) = w(t)\Phi(t, s, \xi).$$

For each ξ let $v(\xi)$ denote a vector perpendicular to $f(y(\xi))$, with $v(0)$ a positive multiple of v_0 and the length of $v(\xi)$ chosen so that $v(\xi)$ is a smooth function of ξ. Let $\psi(t, \xi)$ denote the solution of $\dot{w} = -wDf(x(t, \xi))$ whose value at $t = T(\xi, 0)$ is $v(\xi)^\top$, the transpose of $v(\xi)$:

$$\psi(T(\xi, 0), \xi) = v(\xi)^\top.$$

Theorem 3. *Let*

$$M(\xi) = \int_0^{T(\xi,0)} \psi(t, \xi)\, g(x(t, \xi, 0))\, dt.$$

(The integrand is a row vector times a column vector). If $M(\xi_1) = 0$ and $M'(\xi_1) \neq 0$, then there is a curve $\xi(\mu)$, μ small, with $\xi(0) = \xi_1$, such that $P(\xi(\mu), \mu) = \xi(\mu)$.

To prove the theorem we need to show that $\Delta(\xi, 0) = c(\xi)M(\xi)$ with $c(\xi) \neq 0$. The result then follows from the two lemmas.

Let $\phi(T(\xi, \mu), y(\xi), \mu) = y_0 + \tilde{\xi}(\mu)v_0$, and let $\tilde{P}(\xi, \mu) = v(\xi) \cdot \big(\phi(T(\xi, \mu), y(\xi), \mu) - y_0\big)$. Then

$$\begin{aligned} P(\xi, \mu) = v_0 \cdot \big(\phi(T(\xi, \mu), y(\xi), \mu) - y_0\big) = \tilde{\xi}(\mu) &= \frac{v(\xi) \cdot \tilde{\xi}(\mu)v_0}{v(\xi) \cdot v_0} \\ &= \frac{1}{v(\xi) \cdot v_0}\, v(\xi) \cdot \big(\phi(T(\xi, \mu), y(\xi), \mu) - y_0\big) = \frac{1}{v(\xi) \cdot v_0}\tilde{P}(\xi, \mu) \end{aligned}$$

Therefore

$$\Delta(\xi, 0) = \frac{\partial P}{\partial \mu}(\xi, 0) = \frac{1}{v(\xi) \cdot v_0}\, \frac{\partial \tilde{P}}{\partial \mu}(\xi, 0).$$

Since $v(0)$ is a positive multiple of v_0, we see that for ξ not too big, $\Delta(\xi, 0)$ is a positive multiple of $\frac{\partial \tilde{P}}{\partial \mu}(\xi, 0)$. To complete the proof, we will show that

$$\frac{\partial \tilde{P}}{\partial \mu}(\xi, 0) = M(\xi).$$

From the definition of $\tilde{P}$, we calculate:

$$\frac{\partial \tilde{P}}{\partial \mu}(\xi, 0) = v(\xi) \cdot \left(\frac{\partial \phi}{\partial t}(T(\xi, 0), y(\xi), 0) \frac{\partial T}{\partial \mu}(\xi, 0) + \frac{\partial \phi}{\partial \mu}(T(\xi, 0), y(\xi), 0) \right)$$
$$= v(\xi) \cdot \frac{\partial \phi}{\partial \mu}(T(\xi, 0), y(\xi), 0)$$

because $v(\xi)$ is perpendicular to

$$f(y(\xi)) = \frac{\partial \phi}{\partial t}(0, y(\xi), 0) = \frac{\partial \phi}{\partial t}(T(\xi, 0), y(\xi), 0).$$

Now

$$\frac{\partial \phi}{\partial \mu}(t, y(\xi), 0) = \frac{\partial x}{\partial \mu}(t, \xi, 0)$$

is the solution of the linear differential equation

$$\dot{X} = Df(x(t, \xi, 0))X + g(x(t, \xi, 0)), \quad X(0, \xi) = 0. \tag{1}$$

(ξ is a parameter.) To see this, recall that $x(t, \xi, \mu)$ is the solution of

$$\dot{x} = f(x) + \mu g(x), \quad x(0, \xi, \mu) = y(\xi).$$

Let $x = x(t, \xi, \mu)$ in the differential equation, then differentiate both the differential equation and the initial condition with respect to μ and set $\mu = 0$.

The solution of (**??**) is

$$\frac{\partial x}{\partial \mu}(t, \xi, 0) = X(t, \xi) = \int_0^t \Phi(t, s, \xi) g(x(s, \xi, 0))\, ds.$$

Therefore

$$\frac{\partial \tilde{P}}{\partial \mu}(\xi, 0) = v(\xi) \cdot \frac{\partial \phi}{\partial \mu}(T(\xi, 0), y(\xi), 0) = \psi(T(\xi), \xi) \frac{\partial x}{\partial \mu}(T(\xi), \xi, 0)$$
$$= \psi(T(\xi), \xi) \int_0^{T(\xi)} \Phi(T(\xi), s, \xi) g(x(s, \xi, 0))\, ds = \int_0^{T(\xi)} \psi(s, \xi) g(x(s, \xi, 0))\, ds = M(\xi).$$

This complete the proof.