ANOTHER VIEW OF THE POINCARÉ-LINSTEDT METHOD

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We encountered the Poincaré-Linstedt method as a way of finding a periodic solution of an equation of the form

$$\frac{d^2x}{dt^2} + \epsilon f\left(x, \frac{dx}{dt}\right) + x = 0$$

with $\epsilon > 0$ small. For $\epsilon = 0$ there are many 2π-periodic solution. The method was to look for a solution of period $\frac{2\pi}{\omega(\epsilon)}$ with $\omega(0) = 1$. Such a solution could be written as

$$x(t) = y(\tau), \quad \tau = \omega(\epsilon)\, t,$$

with $y(\tau)$ a 2π-periodic solution of

$$\omega^2 \frac{d^2y}{d\tau^2} + \epsilon f\left(x, \omega\frac{dx}{d\tau}\right) + x = 0.$$

We then expand

$$y(\tau, \epsilon) = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau), \quad \omega(\epsilon) = 1 + \epsilon\omega_0 + \epsilon^2\omega_1 + \ldots,$$

and solve for the y_i and ω_i using the fact that each $y_i(\tau)$ should be 2π-periodic.

Another way to look at this is to consider a system with one more variable,

$$\omega^2 \frac{d^2y}{d\tau^2} + \epsilon f\left(x, \omega\frac{dx}{d\tau}\right) + x = 0,$$

$$\frac{d\omega}{d\tau} = 0,$$

and look for a family of 2π-periodic solutions $(y(\tau, \epsilon), \omega(\epsilon))$ with $\omega(0) = 1$. The expansion is exactly the same, but the cleverness is gone.

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