

Hence, every periodic solution in the interval $[0, a + b + 1]$ is stable. The uniqueness of the periodic solution is a consequence of this result. In fact, the map P is real analytic. Thus, if P has infinitely many fixed points in a compact interval, then P is the identity. This is not true, so P has only a finite number of fixed points. If ξ_0 and ξ_1 are the coordinates of two consecutive fixed points, then the displacement function, that is, $\xi \mapsto P(\xi) - \xi$, has negative slope at two consecutive zeros, in contradiction.

Exercise 1.155. Find an explicit formula for the solution of the differential equation (1.43) and use it to give a direct proof for the existence of a nontrivial periodic solution.

Exercise 1.156. Prove that $P''(\xi) < 0$ for $\xi > 0$, where P is the Poincaré map defined for the differential equation (1.43). Use this result and the inequality (1.45) to prove the uniqueness of the nontrivial periodic solution of the differential equation.

Exercise 1.157. Show that the (stroboscopic) Poincaré map for the differential equation (1.43) has exactly one fixed point on the interval $(0, \infty)$. How many fixed points are there on $(-\infty, 0)$?

Exercise 1.158. Suppose that $h : \mathbb{R} \rightarrow \mathbb{R}$ is a T -periodic function, and $0 < h(t) < 1/4$ for every $t \in \mathbb{R}$. Show that the differential equation $\dot{x} = x(1-x) - h(t)$ has exactly two T -periodic solutions. The differential equation can be interpreted as a model for the growth of a population in a limiting environment that is subjected to periodic harvesting (cf. [200]).

Exercise 1.159. Is it possible for the Poincaré map for a scalar differential equation not to be the identity map on a fixed compact interval and at the same time have infinitely many fixed points in the interval?

Exercise 1.160. [Boundary Value Problem] (a) Prove that the Dirichlet boundary value problem

$$x'' = 1 - x^2, \quad x(0) = 0, \quad x(2) = 0$$

has a solution. Hint: Use the phase plane. Show that the first positive time T such that the orbit with initial conditions $x(0) = 0$ and $x'(0) = 0$ reaches the x -axis is $T < 2$ and for the initial conditions $x(0) = 0$ and $x'(0) = 2/\sqrt{3}$, $T > 2$. To show this fact use the idea in the hint for Exercise 1.12 to construct an integral representation for T . (b) Find a solution of the boundary value problem by shooting and Newton's method (see Exercise 1.124). Hint: Use the phase plane with $x' = y$. Consider the solution $t \mapsto (x(t, \eta), y(t, \eta))$ with initial conditions $x(0) = 0$ and $y(0) = \eta$ and use Newton's method to solve the equation $y(2, \eta) = 0$. Note: The solutions with different choices for the velocity are viewed as shots. The velocity is adjusted until the target is hit.

Exercise 1.161. Consider the linear system

$$\dot{x} = ax, \quad \dot{y} = -by$$

where $a > 0$ and $b > 0$ in the open first quadrant of the phase plane and let ϕ_t denote its flow. (a) Show that $L := \{(\xi, 1) : \xi > 0\}$ and $M := \{(1, \eta) : \eta > 0\}$ are transverse sections for the system. (b) Find a formula for the section map h from L to M . (c) Find a formula for $T : L \rightarrow \mathbb{R}$, called the time-of-flight map, which is defined by $\phi_{T(\xi)}(\xi, 1) = (1, h(\xi))$.

Exercise 1.162. Compute the time required for the solution of the system

$$\dot{x} = x(1-y), \quad \dot{y} = y(x-1)$$

with initial condition $(x, y) = (1, 0)$ to arrive at the point $(x, y) = (2, 0)$. Note that this system has a section map $y \mapsto h(y)$ defined from a neighborhood of $(x, y) = (1, 0)$ on the line given by $x = 1$ to the line given by $x = 2$. Compute $h'(0)$.

Exercise 1.163. Observe that the x -axis is invariant for the system

$$\dot{x} = 1 + xy, \quad \dot{y} = 2xy^2 + y^3,$$

and the trajectory starting at the point $(1, 0)$ crosses the line $x = 3$ at $(3, 0)$. Thus, there is a section map h and a time-of-flight map T from the line $x = 1$ to the line $x = 3$ with both functions defined on some open interval about the point $(1, 0)$ on the line $x = 1$. Compute $T'(0)$ and $h'(0)$.

Exercise 1.164. Research Problem: Consider the second order differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

where f and g are 2π -periodic functions. Determine conditions on f and g that ensure the existence of a periodic solution.

1.9.2 Limit Sets and Poincaré-Bendixson Theory

The general problem of finding periodic solutions for differential equations is still an active area of mathematical research. Perhaps the most well developed theory for periodic solutions is for differential equations defined on the plane. But, even in this case, the theory is far from complete. For example, consider the class of planar differential equations of the form

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

where f and g are quadratic polynomials. There are examples of such "quadratic systems" that have four isolated periodic orbits—"isolated" means that each periodic orbit is contained in an open subset of the plane that contains no other periodic orbits (see Exercise 1.194). But, no one knows at present if there is a quadratic system with more than four isolated periodic orbits. The general question of the number of isolated periodic orbits for a polynomial system in the plane has been open since 1905; it is called Hilbert's 16th problem (see [55], [126], [187], and [197]).

Although there are certainly many difficult issues associated with periodic orbits of planar systems, an extensive theory has been developed that has been successfully applied to help determine the dynamics of many mathematical models. Some of the basic results of this theory will be explained later in this section after we discuss some important general properties of flows of autonomous, not necessarily planar, systems.

The properties that we will discuss enable us to begin to answer the question "What is the long term behavior of a dynamical system?" This is often the most important question about a mathematical model. Ask an engineer what he wants to know about a model ordinary differential equation. Often his response will be the question "What happens if we start the system running and then wait for a long time?" or, in engineering jargon, "What is the steady state behavior of the system?" We already know how to answer these questions in some special circumstances where the steady state behavior corresponds to a rest point or periodic orbit. The following definitions will be used to precisely describe the limiting behavior of an arbitrary orbit.

Definition 1.165. Suppose that ϕ_t is a flow on \mathbb{R}^n and $p \in \mathbb{R}^n$. A point x in \mathbb{R}^n is called an *omega limit point* (ω -limit point) of the orbit through p if there is a sequence of numbers $t_1 \leq t_2 \leq t_3 \leq \dots$ such that $\lim_{i \rightarrow \infty} t_i = \infty$ and $\lim_{i \rightarrow \infty} \phi_{t_i}(p) = x$. The collection of all such omega limit points is denoted $\omega(p)$ and is called the *omega limit set* (ω -limit set) of p . Similarly, the α -limit set $\alpha(p)$ is defined to be the set of all limits $\lim_{i \rightarrow \infty} \phi_{t_i}(p)$ where $t_1 \geq t_2 \geq t_3 \geq \dots$ and $\lim_{i \rightarrow \infty} t_i = -\infty$.

Definition 1.166. The orbit of the point p with respect to the flow ϕ_t is called *forward complete* if $t \rightarrow \phi_t(p)$ is defined for all $t \geq 0$. Also, in this case, the set $\{\phi_t(p) : t \geq 0\}$ is called the *forward orbit* of the point p . The orbit is called *backward complete* if $t \rightarrow \phi_t(p)$ is defined for all $t \leq 0$ and the backward orbit is $\{\phi_t(p) : t \leq 0\}$.

Proposition 1.167. The omega limit set of a point is closed and invariant.

Proof. The empty set is closed and invariant.

Suppose that $\omega(p)$ is not empty for the flow ϕ_t and $x \in \omega(p)$. Consider $\phi_T(x)$ for some fixed $T \in \mathbb{R}$. There is a sequence $t_1 \leq t_2 \leq t_3 \leq \dots$ with $t_i \rightarrow \infty$ and $\phi_{t_i}(p) \rightarrow x$ as $i \rightarrow \infty$. Note that $t_1 + T \leq t_2 + T \leq t_3 + T \leq \dots$ and that $\phi_{t_i+T}(p) = \phi_T(\phi_{t_i}(p))$. By the continuity of the flow, we have that $\phi_T(\phi_{t_i}(p)) \rightarrow \phi_T(x)$ as $i \rightarrow \infty$. Thus, $\phi_T(x) \in \omega(p)$, and therefore $\omega(p)$ is an invariant set.

To show $\omega(p)$ is closed, it suffices to show that $\omega(p)$ is the intersection of closed sets. In fact, we have that

$$\omega(p) = \bigcap_{\tau \geq 0} \text{closure} \{ \phi_t(p) : t \geq \tau \}. \quad \square$$

Proposition 1.168. Suppose that $p \in \mathbb{R}^n$ and the orbit of the flow ϕ_t through the point p is forward complete. If the forward orbit of p has compact closure, then $\omega(p)$ is nonempty, compact, and connected.

Proof. The sequence $\{\phi_n(p)\}_{n=1}^\infty$ is contained in the compact closure of the orbit through p . Thus, it has at least one limit point x . In fact, there is an infinite sequence of integers $n_1 \leq n_2 \leq \dots$ such that $\phi_{n_i}(p) \rightarrow x$ as $i \rightarrow \infty$. Hence, $x \in \omega(p)$, and therefore $\omega(p) \neq \emptyset$.

Since $\omega(p)$ is a closed subset of the compact closure of the orbit through p , the set $\omega(p)$ is compact.

To prove that $\omega(p)$ is connected, suppose to the contrary that there are two disjoint open sets U and V whose union contains $\omega(p)$ such that $\omega(p) \cap U \neq \emptyset$ and $\omega(p) \cap V \neq \emptyset$. There is some $t_1 > 0$ such that $\phi_{t_1}(p) \in U$ and some $t_2 > t_1$ such that $\phi_{t_2}(p) \in V$. But the set $K = \{\phi_t(p) : t_1 \leq t \leq t_2\}$ is the continuous image of an interval, hence a connected set. Thus K cannot be contained in $U \cup V$. In particular, there is at least one $\tau_1 > 0$ such that $\phi_{\tau_1}(p)$ is not in this union.

Similarly we can construct a sequence $\tau_1 \leq \tau_2 \leq \dots$ such that

$$\lim_{i \rightarrow \infty} \tau_i = \infty$$

and for each i the point $\phi_{\tau_i}(p)$ is in the complement of $U \cup V$. By the compactness, the sequence $\{\phi_{\tau_i}(p)\}_{i=1}^\infty$ has a limit point x . Clearly, x is also in $\omega(p)$ and in the complement of $U \cup V$. This is a contradiction. \square

Exercise 1.169. Construct examples to show that the compactness hypothesis of Proposition 1.168 is necessary.

Exercise 1.170. Show that a reparametrization of a flow does not change its omega limit sets. Thus, an omega limit set is determined by an orbit and its direction, not the parametrization of the orbit.

Exercise 1.171. Suppose that x_0 is a rest point for the differential equation $\dot{x} = f(x)$ with flow ϕ_t , and V is a Lyapunov function at x_0 . If, in addition, there is a neighborhood W of the rest point x_0 such that, for each point $p \in W \setminus \{x_0\}$, the function V is not constant on the forward orbit of p , then x_0 is asymptotically stable. Hint: The point x_0 is Lyapunov stable. If it is not asymptotically stable, then there is a point p in the domain of V whose omega limit set $\omega(p)$ is also in the domain of V such that $\omega(p) \neq \{x_0\}$. Show that V is constant on this omega limit set (the constant is the greatest lower bound of the range of V on the forward orbit through p).

Exercise 1.172. Suppose that the differential equation $\dot{x} = f(x)$ with flow ϕ_t has a compact invariant set K , and $V : K \rightarrow \mathbb{R}$ is a continuously differentiable function such that $\dot{V}(x) \leq 0$ for every $x \in K$. If Ω is the largest invariant set in $\{x \in K : \dot{V}(x) = 0\}$, then every solution in K approaches Ω as $t \rightarrow \infty$.

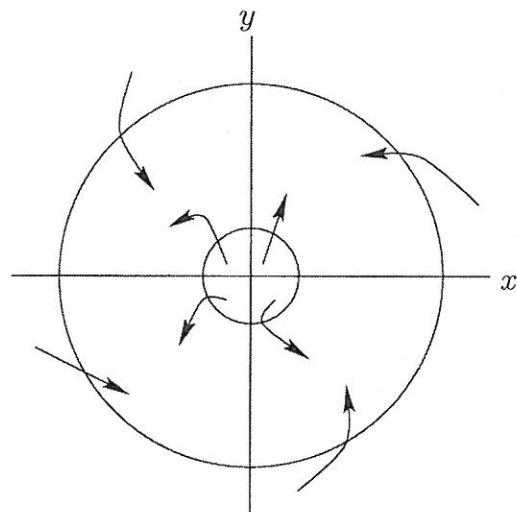


Figure 1.26: A positively invariant annular region for a flow in the plane.

The ω -limit set of a point for a flow in \mathbb{R}^n with $n \geq 3$ can be very complicated; for example, it can be a fractal. But the situation in \mathbb{R}^2 is much simpler. The reason is the deep fact about the geometry of the plane stated in the next theorem.

Theorem 1.173 (Jordan Curve Theorem). *A simple closed (continuous) curve in the plane divides the plane into two connected components, one bounded and one unbounded, each with the curve as boundary.*

Proof. Modern proofs of this theorem use algebraic topology (see for example [212]). \square

This result will play a central role in what follows.

The fundamental result about limit sets for flows of planar differential equations is the Poincaré–Bendixson theorem. There are several versions of this theorem; we will state two of them. The main ingredients of their proofs will be presented later in this section beginning with Lemma 1.187.

Theorem 1.174 (Poincaré–Bendixson). *If Ω is a nonempty compact ω -limit set of a flow in \mathbb{R}^2 , and if Ω does not contain a rest point, then Ω is a periodic orbit.*

A set S that contains the forward orbit of each of its elements is called *positively invariant*. An orbit whose α -limit set is a rest point p and whose ω -limit is a rest point q is said to *connect* p and q . Note: the definition of a connecting orbit allows $p = q$.

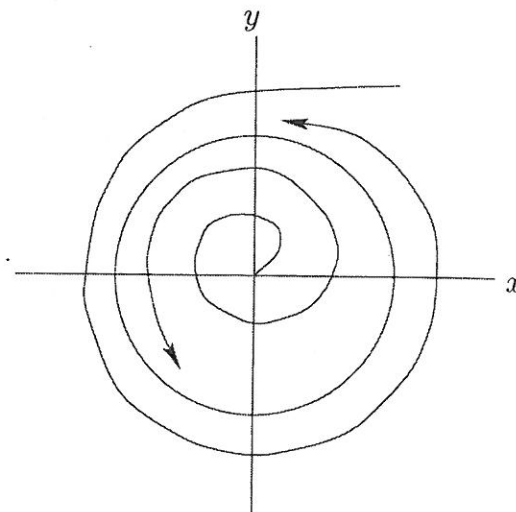


Figure 1.27: A limit cycle in the plane.

Theorem 1.175. *Suppose that ϕ_t is a flow on \mathbb{R}^2 and $S \subseteq \mathbb{R}^2$ is a positively invariant set with compact closure. If $p \in S$ and ϕ_t has at most a finite number of rest points in the closure of S , then $\omega(p)$ is either (i) a rest point, (ii) a periodic orbit, or (iii) a union of finitely many rest points and a nonempty finite or countable infinite set of connecting orbits.*

Exercise 1.176. Illustrate possibility (iii) of the last theorem with an example having an infinite set of connecting orbits.

Exercise 1.177. We have assumed that all flows are smooth. Is this hypothesis required for all the theorems in this section on ω -limit sets?

Definition 1.178. A *limit cycle* Γ is a periodic orbit that is either the ω -limit set or the α -limit set of some point that is in the phase space but not in Γ .

A “conceptual” limit cycle is illustrated in Figure 1.27. In this figure, the limit cycle is the ω -limit set of points in its interior (the bounded component of the plane with the limit cycle removed) and its exterior (the corresponding unbounded component of the plane). A limit cycle that is generated by numerically integrating a planar differential equation is depicted in Figure 1.28 (see [33]).

Sometimes the following alternative definition of a limit cycle is given. A “limit cycle” is an isolated periodic orbit; that is, the unique periodic orbit

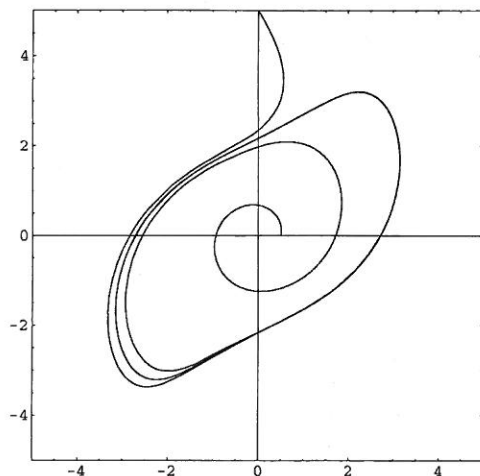


Figure 1.28: Two orbits are numerically computed for the system $\dot{x} = 0.5x - y + 0.1(x^2 - y^2)(x - y)$, $\dot{y} = x + 0.5y + 0.1(x^2 - y^2)(x + y)$: one with initial value $(x, y) = (0.5, 0)$, the other with initial value $(x, y) = (0, 5)$. Both orbits approach a stable limit cycle.

in some open subset of the phase space. This definition is *not* equivalent to Definition 1.178 in general. The two definitions, however, are equivalent for real analytic systems in the plane (see Exercise 1.182).

An annular region is a subset of the plane that is homeomorphic to the closed annulus bounded by the unit circle at the origin and the concentric circle whose radius is two units in length.

The following immediate corollary of the Poincaré–Bendixson theorem is often applied to prove the existence of limit cycles for planar systems.

Theorem 1.179. *If a flow in the plane has a positively invariant annular region S that contains no rest points of the flow, then S contains at least one periodic orbit. If in addition, some point in S is in the forward orbit of a point on the boundary of S , then S contains at least one limit cycle.*

We will discuss two applications of Theorem 1.179 where the main idea is to find a rest-point free annular region as depicted in Figure 1.26.

The first example is provided by the differential equation

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2). \quad (1.46)$$

Note that the annulus S bounded by the circles with radii $\frac{1}{2}$ and 2, respectively, contains no rest points of the system. Let us show that S is positively invariant. To prove this fact, consider the outer normal vector N on ∂S that is the restriction of the vector field $N(x, y) = (x, y, x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ to ∂S and compute the dot product of N with the vector field corresponding to

the differential equation. In fact, the dot product

$$x^2(1 - x^2 - y^2) + y^2(1 - x^2 - y^2) = (x^2 + y^2)(1 - x^2 - y^2)$$

is positive on the circle with radius $\frac{1}{2}$ and negative on the circle with radius 2. Therefore, S is positively invariant and, by Theorem 1.179, there is at least one limit cycle in S .

The differential equation (1.46) is so simple that we can find a formula for its flow. In fact, by changing to polar coordinates (r, θ) , the transformed system

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1$$

decouples, and its flow is given by

$$\phi_t(r, \theta) = \left(\left(\frac{r^2 e^{2t}}{1 - r^2 + r^2 e^{2t}} \right)^{\frac{1}{2}}, \theta + t \right). \quad (1.47)$$

Note that $\phi_t(1, \theta) = (1, \theta + t)$ and, in particular, $\phi_{2\pi}(1, \theta) = (1, \theta + 2\pi)$. Thus, the unit circle in the plane is a periodic orbit with period 2π . Here, of course, we must view θ as being defined modulo 2π , or, better yet, we must view the polar coordinates as coordinates on the cylinder $\mathbb{T} \times \mathbb{R}$ (see Section 1.8.5).

If the formula for the flow (1.47) is rewritten in rectangular coordinates, then the periodicity of the unit circle is evident. In fact, the periodic solution starting at the point $(\cos \theta, \sin \theta) \in \mathbb{R}^2$ (in rectangular coordinates) at $t = 0$ is given by

$$t \mapsto (\tilde{x}(t), \tilde{y}(t)) = (\cos(\theta + t), \sin(\theta + t)).$$

It is easy to see that if $r \neq 0$, then the ω -limit set $\omega((r, \theta))$ is the entire unit circle. Thus, the unit circle is a limit cycle.

If we consider the positive x-axis as a Poincaré section, then we have

$$P(x) = \left(\frac{x^2 e^{4\pi}}{1 - x^2 + x^2 e^{4\pi}} \right)^{\frac{1}{2}}.$$

Here $P(1) = 1$ and $P'(1) = e^{-4\pi} < 1$. In other words, the intersection point of the limit cycle with the Poincaré section is a hyperbolic fixed point of the Poincaré map; that is, the linearized Poincaré map has no eigenvalue on the unit circle of the complex plane. In fact, here the single eigenvalue of the linear transformation of \mathbb{R} given by $x \mapsto P'(1)x$ is inside the unit circle. It should be clear that in this case the limit cycle is an asymptotically stable periodic orbit. We will also call such an orbit a *hyperbolic stable limit cycle*. (The general problem of the stability of periodic orbits is discussed in Chapter 2.)

As a second example of the application of Theorem 1.179, let us consider the very important differential equation

$$\ddot{\theta} + \lambda \dot{\theta} + \sin \theta = \mu$$

where $\lambda > 0$ and μ are constants, and θ is an angular variable; that is, θ is defined modulo 2π . This differential equation is a model for an unbalanced rotor or pendulum with viscous damping $\lambda \dot{\theta}$ and external torque μ .

Consider the equivalent first order system

$$\dot{\theta} = v, \quad \dot{v} = -\sin \theta + \mu - \lambda v, \quad (1.48)$$

and note that, since θ is an angular variable, the natural phase space for this system is the cylinder $\mathbb{T} \times \mathbb{R}$. With this interpretation we will show the following result: *If $|\mu| > 1$, then system (1.48) has a globally attracting limit cycle.* The phrase “globally attracting limit cycle” means that there is a limit cycle Γ on the cylinder and Γ is the ω -limit set of every point on the cylinder. In other words, the steady state behavior of the unbalanced rotor, with viscous damping and sufficiently large torque, is stable periodic motion. (See [143] for the existence of limit cycles in case $|\mu| \leq 1$.)

The system (1.48) with $|\mu| > 1$ has no rest points. (Why?) Also the quantity $-\sin \theta + \mu - \lambda v$ is negative for sufficiently large positive values of v , and it is positive for negative values of v that are sufficiently large in absolute value. Therefore, there are numbers $v_- < 0$ and $v_+ > 0$ such that every forward orbit is contained in the compact subset of the cylinder $A := \{(r, \theta) : v_- \leq v \leq v_+\}$. In addition, A is diffeomorphic to an annular region in the plane. It follows that the Poincaré–Bendixson theorem is valid in A , and therefore the ω -limit set of every point on the cylinder is a limit cycle.

Although there are several ways to prove that the limit cycle is unique, let us consider a proof based on the following propositions: (i) *If the divergence of a vector field is everywhere negative, then the flow of the vector field contracts volume* (see Exercise 2.22). (ii) *Every periodic orbit in the plane surrounds a rest point* (see Exercise 1.189). (A replacement for the first proposition is given in Exercise 1.200; an alternate method of proof is suggested in Exercise 1.202.)

To apply the propositions, note that the divergence of the vector field for system (1.48) is the negative number $-\lambda$. Also, if $|\mu| > 1$, then this system has no rest points. By the second proposition, no periodic orbit of the system is contractable on the cylinder (see panel (a) of Figure 1.29). Thus, if there are two periodic orbits, they must bound an *invariant* annular region on the cylinder as in panel (b) of Figure 1.29. But this contradicts the fact that the area of the annular region is contracted by the flow. It follows that there is a unique periodic orbit on the cylinder that is a globally attracting limit cycle.

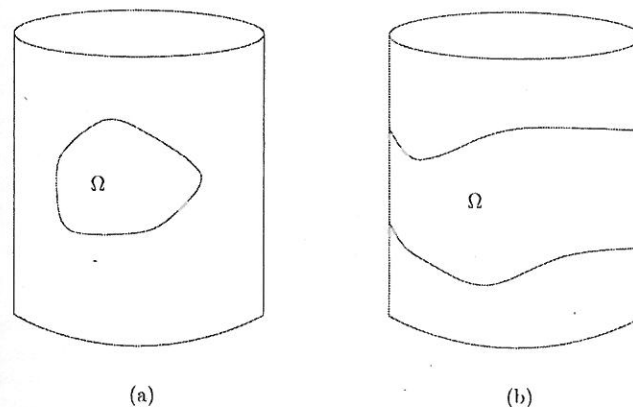


Figure 1.29: Panel (a) depicts a contractable periodic orbit on a cylinder. Note that the region Ω in panel (a) is simply connected. Panel (b) depicts two periodic orbits that are not contractable; they bound a multiply connected region Ω on the cylinder.

Exercise 1.180. Give a direct proof that the point $(1/\sqrt{2}, 1/\sqrt{2})$ on the unit circle is an ω -limit point of the point $(3, 8)$ for the flow of system (1.46).

Exercise 1.181. Discuss the phase portrait of system (1.48) for $|\mu| < 1$.

Exercise 1.182. (a) Show that the set containing “limit cycles” defined as isolated periodic orbits is a proper subset of the set of limit cycles. Also, if the differential equation is a real analytic planar autonomous system, then the two concepts are the same. Hint: Imagine an annular region consisting entirely of periodic orbits. The boundary of the annulus consists of two periodic orbits that might be limit cycles, but neither of them is isolated. To prove that an isolated periodic orbit Γ is a limit cycle, show that every section of the flow at a point $p \in \Gamma$ has a subset that is a Poincaré section at p . For an analytic system, again consider a Poincaré section and the associated Poincaré map P . Zeros of the analytic displacement function $\xi \mapsto P(\xi) - \xi$ correspond to periodic orbits. (b) Show that the polynomial (hence real analytic) system in \mathbb{R}^3 given by

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2), \\ \dot{y} &= x + y(1 - x^2 - y^2), \\ \dot{z} &= 1 - x^2 - y^2 \end{aligned} \quad (1.49)$$

has limit cycles that are not isolated. (c) Determine the long-term behavior of the system (1.49). In particular, show that

$$\lim_{t \rightarrow \infty} z(t) = z(0) - \frac{1}{2} \ln(x^2(0) + y^2(0)).$$

Exercise 1.183. Show that the system

$$\dot{x} = ax - y + xy^2, \quad \dot{y} = x + ay + y^3$$

has an unstable limit cycle for $a < 0$ and no limit cycle for $a > 0$. Hint: Change to polar coordinates.

Exercise 1.184. Show that the system

$$\begin{aligned}\dot{x} &= y + x(x^2 + y^2 - 1) \sin \frac{1}{x^2 + y^2 - 1}, \\ \dot{y} &= -x + y(x^2 + y^2 - 1) \sin \frac{1}{x^2 + y^2 - 1}\end{aligned}$$

has infinitely many limit cycles in the unit disk.

Exercise 1.185. Prove: An analytic planar system cannot have infinitely many limit cycles that accumulate on a periodic orbit. Note: This (easy) exercise is a special case of a deep result: An analytic planar system cannot have infinitely many limit cycles in a compact subset of the plane; and, a polynomial system cannot have infinitely many limit cycles (see [79] and [126]).

Exercise 1.186. Consider the differential equation

$$\dot{x} = -ax(x^2 + y^2)^{-1/2}, \quad \dot{y} = -ay(x^2 + y^2)^{-1/2} + b$$

where a and b are positive parameters. The model represents the flight of a projectile, with speed a and heading toward the origin, that is moved off course by a constant force with strength b . Determine conditions on the parameters that ensure the solution starting at the point $(x, y) = (p, 0)$, for $p > 0$, reaches the origin. Hint: Change to polar coordinates and study the phase portrait of the differential equation on the cylinder. Explain your result geometrically. The differential equation is not defined at the origin. Is this a problem?

The next two lemmas are used in the proof of the Poincaré Bendixson theorem. The first lemma is a corollary of the Jordan curve theorem.

Lemma 1.187. If Σ is a section for the flow ϕ_t and if $p \in \mathbb{R}^2$, then the orbit through the point p intersects Σ in a monotone sequence; that is, if $\phi_{t_1}(p)$, $\phi_{t_2}(p)$, and $\phi_{t_3}(p)$ are on Σ and if $t_1 < t_2 < t_3$, then $\phi_{t_2}(p)$ lies strictly between $\phi_{t_1}(p)$ and $\phi_{t_3}(p)$ on Σ or $\phi_{t_1}(p) = \phi_{t_2}(p) = \phi_{t_3}(p)$.

Proof. The proof is left as an exercise. Hint: Reduce to the case where t_1 , t_2 , and t_3 correspond to consecutive crossing points. Then, consider the curve formed by the union of $\{\phi_t(p) : t_1 \leq t \leq t_2\}$ and the subset of Σ between $\phi_{t_1}(p)$ and $\phi_{t_2}(p)$. Draw a picture. \square

Lemma 1.188. If Σ is a section for the flow ϕ_t and if $p \in \mathbb{R}^2$, then $\omega(p) \cap \Sigma$ contains at most one point.

Proof. The proof is by contradiction. Suppose that $\omega(p) \cap \Sigma$ contains at least two points, x_1 and x_2 . By rectification of the flow at x_1 and at x_2 , that is, by the rectification lemma (Lemma 1.120), it is easy to see that there are sequences $\{\phi_{t_i}(p)\}_{i=1}^\infty$ and $\{\phi_{s_i}(p)\}_{i=1}^\infty$ in Σ such that $\lim_{i \rightarrow \infty} \phi_{t_i}(p) = x_1$ and $\lim_{i \rightarrow \infty} \phi_{s_i}(p) = x_2$. By the rectification lemma in Exercise 1.125,

such sequences can be found in Σ . Indeed, we can choose the rectifying neighborhood so that the image of the Poincaré section is a line segment transverse to the rectified flow. In this case, it is clear that if an orbit has one of its points in the rectifying neighborhood, then this orbit passes through the Poincaré section.

By choosing a local coordinate on Σ , let us assume that Σ is an open interval. Working in this local chart, there are open subintervals J_1 at x_1 and J_2 at x_2 such that $J_1 \cap J_2 = \emptyset$. Moreover, by the definition of limit sets, there is an integer m such that $\phi_{t_m}(p) \in J_1$; an integer n such that $s_n > t_m$ and $\phi_{s_n}(p) \in J_2$; and an integer ℓ such that $t_\ell > s_n$ and $\phi_{t_\ell}(p) \in J_1$. By Lemma 1.187, the point $\phi_{s_n}(p)$ must be between the points $\phi_{t_m}(p)$ and $\phi_{t_\ell}(p)$ on Σ . But this is impossible because the points $\phi_{t_m}(p)$ and $\phi_{t_\ell}(p)$ are in J_1 , whereas $\phi_{s_n}(p)$ is in J_2 . \square

We are now ready to prove the Poincaré-Bendixson theorem (Theorem 1.174): If Ω is a nonempty compact ω -limit set of a flow in \mathbb{R}^2 , and if Ω does not contain a rest point, then Ω is a periodic orbit.

Proof. Suppose that $\omega(p)$ is nonempty, compact, and contains no rest points. Choose a point $q \in \omega(p)$. We will show first that the orbit through q is closed.

Consider $\omega(q)$. Note that $\omega(q) \subseteq \omega(p)$ and $\omega(q)$ is not empty. (Why?) Let $x \in \omega(q)$. Since x is not a rest point, there is a section Σ at x and a sequence on Σ consisting of points on the orbit through q that converges to x . These points are in $\omega(p)$. But, by the last corollary, this is impossible unless every point in this sequence is the point x . Since q is not a rest point, this implies that q lies on a closed orbit Γ , as required. In particular, the limit set $\omega(p)$ contains the closed orbit Γ .

To complete the proof we must show $\omega(p) \subseteq \Gamma$. If $\omega(p) \neq \Gamma$, then we will use the connectedness of $\omega(p)$ to find a sequence $\{p_n\}_{n=1}^\infty \subset \omega(p) \setminus \Gamma$ that converges to a point z on Γ . To do this, consider the union A_1 of all open balls with unit radius centered at some point in Γ . The set $A_1 \setminus \Gamma$ must contain a point in $\omega(p)$. If not, consider the union $A_{1/2}$, (respectively $A_{1/4}$) of all open balls with radius $\frac{1}{2}$ (respectively $\frac{1}{4}$) centered at some point in Γ . Then the set $A_{1/4}$ together with the complement of the closure of $A_{1/2}$ “disconnects” $\omega(p)$, in contradiction. By repeating the argument with balls whose radii tend to zero, we can construct a sequence of points in $\omega(p) \setminus \Gamma$ whose distance from Γ tends to zero. Using the compactness of $\omega(p)$, there is a subsequence, again denoted by $\{p_n\}_{n=1}^\infty$, in $\omega(p) \setminus \Gamma$ that converges to a point $z \in \Gamma$.

Let U denote an open set at z such that the flow is rectified in a diffeomorphic image of U . There is some integer n such that $p_n \in U$. But, by using the rectification lemma, it is easy to see that the orbit through p_n has a point y of intersection with some Poincaré section Σ at z . Because

p_n is not in Γ , the points y and z are distinct elements of the set $\omega(p) \cap \Sigma$, in contradiction to Lemma 1.188. \square

Exercise 1.189. Suppose that γ is a periodic orbit of a smooth flow defined on \mathbb{R}^2 . Use Zorn's lemma to prove that γ surrounds a rest point of the flow. That is, the bounded component of the plane with the periodic orbit removed contains a rest point. Note: See Exercise 1.217 for an alternative proof.

Exercise 1.190. Use Exercise 1.189 to prove Brouwer's fixed point theorem for the closed unit disk \mathbb{D} in \mathbb{R}^2 . Hint: First prove the result for a smooth function $f: \mathbb{D} \rightarrow \mathbb{D}$ by considering the vector field $f(x) - x$, and then use the following result: A continuous transformation of \mathbb{D} is the uniform limit of smooth transformations [123, p. 253].

Exercise 1.191. Suppose that a closed ball in \mathbb{R}^n is positively invariant under the flow of an autonomous differential equation on \mathbb{R}^n . Prove that the ball contains a rest point or a periodic orbit. Hint: Apply Brouwer's fixed point theorem to the time-one map of the flow. Explain the differences between this result and the Poincaré-Bendixson theorem.

Exercise 1.192. Construct an example of an (autonomous) differential equation defined on all of \mathbb{R}^3 that has an (isolated) limit cycle but no rest points.

Exercise 1.193. Prove: A nonempty ω -limit set of an orbit of a gradient system consists entirely of rest points.

Exercise 1.194. Is a limit cycle isolated from all other periodic orbits? Hint: Consider planar vector fields of class C^1 and those of class C^ω —real analytic vector fields. Study the Poincaré map on an associated transversal section.

The next theorem can often be used to show that no periodic orbits exist.

Proposition 1.195 (Dulac's Criterion). *Consider a smooth differential equation on the plane*

$$\dot{x} = g(x, y), \quad \dot{y} = h(x, y).$$

If there is a smooth function $B(x, y)$ defined on a simply connected region $\Omega \subseteq \mathbb{R}^n$ such that the quantity $(Bg)_x + (Bh)_y$ is not identically zero and of fixed sign on Ω , then there are no periodic orbits in Ω .

Proof. We will prove Bendixson's criterion, which is the special case of the theorem where $B(x, y) \equiv 1$ (see Exercise 1.198 for the general case). In other words, we will prove that if the divergence of $f := (g(x, y), h(x, y))$ given by

$$\operatorname{div} f(x, y) := g_x(x, y) + h_y(x, y)$$

is not identically zero and of fixed sign in a simply connected region Ω , then there are no periodic orbits in Ω .

Suppose that Γ is a closed orbit in Ω and let G denote the bounded region of the plane bounded by Γ . Note that the line integral of the one form $g dy - h dx$ over Γ vanishes. (Why?) On the other hand, by Green's theorem, the integral can be computed by integrating the two-form $(\operatorname{div} f) dx dy$ over G . Since, by the hypothesis, the divergence of f does not vanish, the integral of the two-form over G does not vanish, in contradiction. Thus, no such periodic orbit can exist. \square

The function B mentioned in the last proposition is called a Dulac function.

We end this section with a result about global asymptotic stability in the plane.

Theorem 1.196. *Consider a smooth differential equation on the plane*

$$\dot{x} = g(x, y), \quad \dot{y} = h(x, y)$$

that has the origin as a rest point. Let J denote the Jacobian matrix for the transformation $(x, y) \mapsto (g(x, y), h(x, y))$, and let ϕ_t denote the flow of the differential equation. If the following three conditions are satisfied, then the origin is globally asymptotically stable.

- Condition 1. For each $(x, y) \in \mathbb{R}^2$, the trace of J given by $g_x(x, y) + h_y(x, y)$ is negative.*
- Condition 2. For each $(x, y) \in \mathbb{R}^2$, the determinant of J given by $g_x(x, y)h_y(x, y) - g_y(x, y)h_x(x, y)$ is positive.*
- Condition 3. For each $(x, y) \in \mathbb{R}^2$, the forward orbit $\{\phi_t(x, y) : 0 \leq t < \infty\}$ is bounded.*

Proof. From the hypotheses on the Jacobian matrix, if there is a rest point, the eigenvalues of its associated linearization all have negative real parts. Therefore, each rest point is a hyperbolic attractor; that is, the basin of attraction of the rest point contains an open neighborhood of the rest point. This fact follows from Hartman's theorem (Theorem 1.47) or Theorem 2.61. In particular, the origin is a hyperbolic attractor.

By the hypotheses, the trace of the Jacobian (the divergence of the vector field) is negative over the entire plane. Thus, by Bendixson's criterion, there are no periodic solutions.

Let Ω denote the basin of attraction of the origin. Using the continuity of the flow, it is easy to prove that Ω is open. In addition, it is easy to prove that the boundary of Ω is closed and contains no rest points.

We will show that the boundary of Ω is positively invariant. If not, then there is a point p in the boundary and a time $T > 0$ such that either $\phi_T(p)$ is in Ω or such that $\phi_T(p)$ is in the complement of the closure of Ω in the plane. In the first case, since $\phi_T(p)$ is in Ω , it is clear that $p \in \Omega$, in contradiction. In the second case, there is an open set V in the complement of the closure of Ω that contains $\phi_T(p)$. The inverse image of V under the continuous map ϕ_T is an open set U containing the boundary point p . By

the definition of boundary, U contains a point $q \in \Omega$. But then, q is mapped to a point in the complement of the closure of Ω , in contradiction to the fact that q is in the basin of attraction of the origin.

If the boundary of Ω is not empty, consider one of its points. The (bounded) forward orbit through the point is precompact and contained in the (closed) boundary of Ω . Thus, its ω -limit set is contained in the boundary of Ω . Since the boundary of Ω contains no rest points, an application of the Poincaré–Bendixson theorem shows this ω -limit set is a periodic orbit, in contradiction. Thus, the boundary is empty and Ω is the entire plane. \square

Theorem 1.196 is a (simple) special case of the “Markus–Yamabe problem.” In fact, the conclusion of the theorem is true without assuming Condition 3 (see [104]).

Exercise 1.197. Prove: If $\delta > 0$, then the origin is a global attractor for the system

$$\dot{u} = (u - v)^3 - \delta u, \quad \dot{v} = (u - v)^3 - \delta v.$$

Also, the origin is a global attractor of orbits in the first quadrant for the system

$$\dot{u} = uv(u - v)(u + 1) - \delta u, \quad \dot{v} = vu(v - u)(v + 1) - \delta v.$$

(Both of these first order systems are mentioned in [229].)

Exercise 1.198. [Dulac’s Criterion] (a) Prove Proposition 1.195. (b) Use Dulac’s criterion to prove a result due to Nikolai N. Bautin: The system

$$\dot{x} = x(a + bx + cy), \quad \dot{y} = y(\alpha + \beta x + \gamma y)$$

has no limit cycles. Hint: Show that no periodic orbit crosses a coordinate axis. Reduce the problem to showing that there are no limit cycles in the first quadrant. Look for a Dulac function of the form $x^r y^s$. After some algebra the problem reduces to showing that a certain two-parameter family of lines always has a member that does not pass through the (open) first quadrant.

Exercise 1.199. (a) Suppose that the system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ has a periodic orbit Γ with period T and B is a positive real valued function defined on some open neighborhood of Γ (as in Dulac’s Criterion). Prove that Γ is a periodic orbit of the system $\dot{x} = B(x, y)f(x, y)$, $\dot{y} = B(x, y)g(x, y)$ with period

$$\tau = \int_0^T \frac{1}{B(x(t), y(t))} ds$$

where $t \mapsto (x(t), y(t))$ is a periodic solution of the original system whose orbit is Γ . (b) How does the period of the limit cycle of system (1.46) change if its vector field is multiplied by $(1 + x^2 + y^2)^\alpha$? Hint: The solution ρ of the initial value problem

$$\dot{\rho} = B((x(\rho), y(\rho))), \quad \rho(0) = 0$$

satisfies the identity $\rho(t + \tau) = \rho(t) + T$.

Exercise 1.200. [Uniqueness of Limit Cycles] (a) Prove the following proposition: If the divergence of a plane vector field is of fixed sign in an annular region Ω of the plane, then the associated differential equation has at most one periodic orbit in Ω . Hint: Use Green’s theorem. (b) Recall Dulac’s criterion from Exercise 1.198 and note that if the divergence of the plane vector field F is not of fixed sign in Ω , then it might be possible to find a nonnegative function $B : \Omega \rightarrow \mathbb{R}$ such that the divergence of BF does have fixed sign in Ω . As an example, consider the van der Pol oscillator,

$$\dot{x} = y, \quad \dot{y} = -x + \lambda(1 - x^2)y$$

and the “Dulac function” $B(x, y) = (x^2 + y^2 - 1)^{-1/2}$. Show that van der Pol’s system has at most one limit cycle in the plane. (The remarkable Dulac function B was discovered by L. A. Cherkas.) (c) Can you prove that the van der Pol oscillator has at least one limit cycle in the plane? Hint: Change coordinates using the Liénard transformation

$$u = x, \quad v = y - \lambda(x - \frac{1}{3}x^3)$$

to obtain the Liénard system

$$\dot{u} = v + \lambda(u - \frac{1}{3}u^3), \quad \dot{v} = -u.$$

In Chapter 5 we will prove that the van der Pol system has a limit cycle if $\lambda > 0$ is sufficiently small. In fact, this system has a limit cycle for each $\lambda > 0$. For this result, and for more general results about limit cycles of the important class of planar systems of the form

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x),$$

see [101, p. 154], [123, p. 215], [141, p. 267], and [183, p. 250].

Exercise 1.201. (a) Prove that the system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

has a unique globally attracting limit cycle on the punctured plane. (b) Find all rest points of the system

$$\dot{x} = x - y - x^n, \quad \dot{y} = x + y - y^n,$$

where n is a positive odd integer and determine their stability. (c) Prove that the system has a unique stable limit cycle. (d) What is the limiting shape of the limit cycle as $n \rightarrow \infty$?

Exercise 1.202. Show there is a unique limit cycle for system (1.48) with $|u| > 1$ by proving the existence of a fixed point for a Poincaré map and by proving that every limit cycle is stable. Hint: Recall the analysis of system (1.44) and consider $dv/d\theta$.

Exercise 1.203. Can a system of the form

$$\dot{x} = y, \quad \dot{y} = f(x) - af'(x)y,$$

where f is a smooth function and a is a parameter, have a limit cycle? Hint: Consider a Liénard transformation.

Exercise 1.204. Draw the phase portrait of the system

$$\dot{x} = y + 2x(1 - x^2 - y^2), \quad \dot{y} = -x.$$

Exercise 1.205. [Rigid Body Motion] The Euler equations for rigid body motion are presented in Exercise 1.77. Recall that the momentum vector is given by $M = A\Omega$ where A is a symmetric matrix and Ω is the angular velocity vector, and Euler's equation is given by $\dot{M} = M \times \Omega$. For ν a positive definite symmetric matrix and F a constant vector, consider the differential equation

$$\dot{M} = M \times \Omega + F - \nu M.$$

Here, the function $M \mapsto \nu M$ represents viscous friction and F is the external force (see [16]). Prove that all orbits of the differential equation are bounded, and therefore every orbit has a compact ω -limit set.

Exercise 1.206. (a) Prove that the origin is a center for the system $\ddot{x} + \dot{x} + x = 0$. (b) Show that this system has unbounded orbits. (c) Describe the boundary between the bounded and unbounded orbits?

Exercise 1.207. Draw the phase portrait for the system $\ddot{x} = x^2 - x^3$. Is the solution with initial conditions $x(0) = \frac{1}{2}$ and $\dot{x}(0) = 0$ periodic?

Exercise 1.208. Draw the phase portrait of the Hamiltonian system $\ddot{x} + x - x^2 = 0$. Give an explicit formula for the Hamiltonian and use it to justify the features of the phase portrait.

Exercise 1.209. Let $t \mapsto x(t)$ denote the solution of the initial value problem

$$\ddot{x} + \dot{x} + x + x^3 = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Determine $\lim_{t \rightarrow \infty} x(t)$.

Exercise 1.210. Show that the system

$$\dot{x} = x - y - (x^2 + \frac{3}{2}y^2)x, \quad \dot{y} = x + y - (x^2 + \frac{1}{2}y^2)y$$

has a unique limit cycle.

Exercise 1.211. Find the rest points in the phase plane of the differential equation $\ddot{x} + (x^2 + x^2 - 1)\dot{x} + x = 0$ and determine their stability. Also, show that the system has a unique stable limit cycle.

Exercise 1.212. Determine the ω -limit set of the solution of the system

$$\dot{x} = 1 - x + y^3, \quad \dot{y} = y(1 - x + y)$$

with initial condition $x(0) = 10$, $y(0) = 0$.

Exercise 1.213. Show that the system

$$\dot{x} = -y + xy, \quad \dot{y} = x + \frac{1}{2}(x^2 - y^2)$$

has periodic solutions, but no limit cycles.

Exercise 1.214. Consider the van der Pol equation

$$\ddot{x} + (x^2 - \epsilon)\dot{x} + x = 0,$$

where ϵ is a real parameter. How does the stability of the trivial solution change with ϵ . Show that the van der Pol equation has a unique stable limit cycle for $\epsilon = 1$. What would you expect to happen to this limit cycle as ϵ shrinks to $\epsilon = 0$. What happens for $\epsilon < 0$?

Exercise 1.215. Find an explicit nonzero solution of the differential equation

$$t^2 x^2 \ddot{x} + \dot{x} = 0.$$

Define new variables $u = 2(3tx^2)^{-1/2}$, $v = -4\dot{x}(3x^3)^{-1/2}$ and show that

$$\frac{dv}{du} = \frac{3v(v - u^2)}{2u(v - u)}.$$

Draw the phase portrait of the corresponding first order system

$$\dot{u} = 2u(v - u), \quad \dot{v} = 3v(v - u^2).$$

Exercise 1.216. [Yorke's Theorem] A theorem of James Yorke states that if $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz on the open set U with Lipschitz constant L and Γ is a periodic orbit of $\dot{x} = f(x)$ contained in U , then the period of Γ is larger than $2\pi/L$ (see [238]). Use Yorke's theorem to estimate a lower bound for the period of the limit cycle solution of the system in Exercise 1.201 part (a). Note: The period of the periodic orbit is approximately 7.5. Hint: Use the mean value theorem and note that the norm of a matrix (with respect to the usual Euclidean norm) is the square root of the spectral radius of the matrix transpose times the matrix (that is; $\|A\| = \sqrt{\rho(A^T A)}$).

Exercise 1.217. [Poincaré index] Let C be a simple closed curve not passing through a rest point of the vector field X in the plane with components (f, g) . Define the Poincaré index of X with respect to C to be

$$I(X, C) = \frac{1}{2\pi} \int_C d \arctan \left(\frac{g}{f} \right);$$

it is the total change the angle $(f(x, y), g(x, y))$ makes with respect to the (positive) x -axis as (x, y) traverses C exactly once counter clockwise (see, for example, [59] or [141]). (a) Prove: The index is an integer. (b) Prove: The index does not change with a deformation of C (as long as the deformed curve does not pass through a rest point). (c) Prove: If C is smooth and T is a continuous choice of the tangent vector along this curve, then $I(T, C) = 1$. In particular, the index of a vector field with respect to one of its closed orbits is unity. (d) The index of a point with respect to X is defined to be the index of X with respect to an admissible curve C that surrounds this point and no other rest point of X . Prove: The index of a regular point (a point that is not a rest point) is zero. (e) Prove: A periodic orbit surrounds at least one rest point.