ADJOINT EQUATION AND MELNIKOV FUNCTION

STEVE SCHECTER

1. Adjoint equation

Consider the linear differential equation $\dot{x} = A(t)x$ ($x = n \times 1$ column vector, $A = n \times n$ matrix). The *adjoint equation* is $\dot{y} = -A(t)^T y$ ($y = n \times 1$ column vector) or equivalently $\dot{w} = -wA(t)$ ($w = 1 \times n$ row vector). I'll use the second form.

Proposition 1.1. *If $x(t)$ is a solution of $\dot{x} = A(t)x$ and $w(t)$ is a solution of $\dot{w} = -wA(t)$, the $w(t)x(t)$ (row vector times column vector) is constant.*

Proof.

$$\frac{d}{dt}w(t)x(t) = \dot{w}x + w\dot{x} = -wAx + wAx = 0.$$

$\square$

Proposition 1.2. *If $\Phi(t)$ is a fundamental matrix solution of $\dot{x} = A(t)x$, then $\Phi^{-1}(t)$ is a fundamental matrix solution of $\dot{w} = -wA(t)$. (It's rows are linearly independent solutions.)*

Proof.

$$\begin{aligned} 0 = \frac{d}{dt}I = \frac{d}{dt}\left(\Phi^{-1}(t)\Phi(t)\right) &= \left(\frac{d}{dt}\Phi^{-1}(t)\right)\Phi(t) + \Phi^{-1}(t)\left(\frac{d}{dt}\Phi(t)\right) \\ &= \left(\frac{d}{dt}\Phi^{-1}(t)\right)\Phi(t) + \Phi^{-1}(t)A(t)\Phi(t) \Rightarrow \frac{d}{dt}\Phi^{-1}(t) = -\Phi^{-1}(t)A(t). \end{aligned}$$

Therefore the rows of $\Phi^{-1}(t)$ are solutions of $\dot{w} = -wA(t)$, and of course they are linearly independent. $\square$

Corollary 1.3. *For $n = 2$, let*

$$\Phi(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

be a fundamental matrix solution of $\dot{x} = A(t)x$. Then a fundamental matrix solution of $\dot{w} = -wA(t)$ is

$$\Phi^{-1}(t) = \frac{1}{a(t)d(t) - b(t)c(t)} \begin{pmatrix} d(t) & -b(t) \\ -c(t) & a(t) \end{pmatrix}.$$

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The point of the Corollary is that for $n = 2$, if you know just one solution of $\dot{x} = A(t)x$, i.e., one column of $\Phi(t)$, and you know $\det \Phi(t)$ (which can be calculated using Liouville's formula), then you know one row of $\Phi^{-1}(t)$, i.e., one solution of $\dot{w} = -wA(t)$.

Recall that the state transition matrix of $\dot{x} = A(t)x$ is $\Phi(t,s)$, with the property that for any solution, $x(t) = \Phi(t,s)x(s)$. Similarly, the state transition matrix of $\dot{w} = -wA(t)$ is $\Psi(t,s)$, with the property that for any solution, $w(t)\Psi(t,s) = w(s)$.

Proposition 1.4. $\Psi(t,s) = \Phi(t,s)$.

Proof. Fix t. $\Psi(t,t)\Phi(t,t) = I \cdot I = I$. Also

$$\begin{aligned}\frac{\partial}{\partial s}\left(\Psi(t,s)\Phi(s,t)\right) &= \left(\frac{\partial}{\partial s}\Psi(t,s)\right)\Phi(s,t) + \Psi(t,s)\left(\frac{\partial}{\partial s}\Phi(s,t)\right) \\ &= -\Psi(t,s)A(s)\Phi(s,t) + \Psi(t,s)A(s)\Phi(s,t) = 0.\end{aligned}$$

Therefore, as a function of s with t fixed, $\Psi(t,s)\Phi(s,t) \equiv I$, so $\Psi(t,s) = \Phi^{-1}(s,t) = \Phi(t,s)$. □

2. Melnikov integral

Consider a differential equation $\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^2$, $\mu \in \mathbb{R}$, with two hyperbolic saddles $p_-(\mu)$ and $p_+(\mu)$. We assume that for $\mu = 0$ there is a solution $x_*(t)$ with $\lim_{t\to-\infty} x_*(t) = p_-(0)$ and $\lim_{t\to\infty} x_*(t) = p_+(0)$. We are interested in whether this connection between equilibria breaks as the parameter μ changes.

Let $x_*(0) = x_0$. The velocity vector of $x_*(t)$ at $t = 0$ is

$$\dot{x}_*(0) = f(x_0, 0) = \left(f_1(x_0,0), f_2(x_0,0)\right).$$

Let

$$u_0 = \frac{1}{\|f(x_0,0)\|^2}\left(-f_2(x_0,0), f_1(x_0,0)\right),$$

which is orthogonal to $f(x_0,0)$. Consider a line segment Σ through x_0 in the direction u_0; Σ is given as $x = x_0(\xi) = x_0 + \xi u_0$, $|\xi| < \alpha$. (Vectors in this paragraph are column vectors.)

Let $x_-(t,\mu)$ be a family of solutions of $\dot{x} = f(x,\mu)$ with

(1) $x_-(0,\mu) \in \Sigma$,
(2) $\lim_{t\to-\infty} x_-(t,\mu) = p_-(\mu)$, and
(3) $x_-(t,0) = x_*(t)$.

These solutions lie in the unstable manifold of $p_-(\mu)$.

Similarly, let $x_+(t,\mu)$ be a family of solutions of $\dot{x} = f(x,\mu)$ with

(1) $x_+(0,\mu) \in \Sigma$,
(2) $\lim_{t\to\infty} x_+(t,\mu) = p_+(\mu)$, and
(3) $x_+(t,0) = x_*(t)$.

These solutions lie in the stable manifold of $p_+(\mu)$.

We have

$$x_-(0,\mu) = x_0 + \xi_-(\mu)u_0, \quad x_+(0,\mu) = x_0 + \xi_+(\mu)u_0, \quad \xi_-(0) = \xi_+(0) = 0.$$

We define the separation function

$$S(\mu) = \xi_-(\mu) - \xi_+(\mu).$$

If $S(\mu) = 0$, then there is a solution of $\dot{x} = f(x,\mu)$ that goes from $p_-(\mu)$ to $p_+(\mu)$. In other words, the unstable manifold of $p_-(\mu)$ meets the stable manifold of $p_+(\mu)$. We have $S(0) = 0$, and we want to calculate $S'(0) = \xi_-'(0) - \xi_+'(0)$.

Let $\psi_0 = \begin{pmatrix} -f_2(x_0) & f_1(x_0) \end{pmatrix}$ (a row vector). Then

$$\frac{d\xi_\pm}{d\mu}(0) = \psi_0 \frac{d\xi_\pm}{d\mu}(0)u_0 = \psi_0 \frac{\partial x_\pm}{\partial \mu}(0,0).$$

We calculate:

$$\frac{\partial^2 x_-}{\partial t \partial \mu}(t,\mu) = \frac{\partial^2 x_-}{\partial \mu \partial t}(t,\mu) = \frac{\partial}{\partial \mu} f(x_-(t,\mu),\mu) = D_x f(x_-(t,\mu),\mu)\frac{\partial x_-}{\partial \mu}(t,\mu) + \frac{\partial f}{\partial \mu}(x_-(t,\mu),\mu).$$

Setting $\mu = 0$ we find that $\frac{\partial x_-}{\partial \mu}(t,0)$ satisfies the inhomogeneous linear differential equation

$$\dot{v} = D_x f(x_*(t),0)v + \frac{\partial f}{\partial \mu}(x_*(t),0). \tag{1}$$

Let the state transition matrix of the homogeneous linear differential equation $\dot{v} = D_x f(x_*(t),0)v$ be $\Phi(t,s)$. By the variation of parameters formula, the solution of (1) with $v(-T)$ given is

$$v(t) = \Phi(t,-T)v(-T) + \int_{-T}^{t} \Phi(t,s)\frac{\partial f}{\partial \mu}(x_*(s),0)ds.$$

Let $\psi(t)$ denote the solution of the adjoint equation $\dot{w} = -wD_x f(x_*(t),0)$ with initial condition $\psi(0) = \psi_0$. Then

$$\begin{aligned} \frac{d\xi_-}{d\mu}(0) = \psi(0)\frac{\partial x_-}{\partial \mu}(0,0) &= \psi(0)\left(\Phi(0,-T)\frac{\partial x_-}{\partial \mu}(-T,0) + \int_{-T}^{0} \Phi(0,s)\frac{\partial f}{\partial \mu}(x_*(s),0)ds\right) \\ &= \psi(-T)\frac{\partial x_-}{\partial \mu}(-T,0) + \int_{-T}^{0} \psi(s)\frac{\partial f}{\partial \mu}(x_*(s),0)ds. \end{aligned} \tag{2}$$

Proposition 2.1. $\psi(t) = \exp\left(-\int_0^t \operatorname{div} f(x_*(s),0)ds\right)\begin{pmatrix} -\dot{x}_{*2}(t) & \dot{x}_{*1}(t) \end{pmatrix}$.

Proof. One solution of the linear differential equation $\dot{v} = D_x f(x_*(t),0)v$ is $\dot{x}_*(t) = (\dot{x}_{*1}, \dot{x}_{*2})$. Suppose $\begin{pmatrix} \dot{x}_*(t) & v(t) \end{pmatrix}$ is a fundamental matrix solution of $\dot{v} = D_x f(x_*(t),0)v$ with determinant 1 at $t = 0$. By Liouville's Formula, the determinant at time t is

$$\exp\left(\int_0^t \operatorname{tr} D_x f(x_*(s),0)ds\right) = \exp\left(\int_0^t \operatorname{div} f(x_*(s),0)ds\right).$$

The proposition then follows from Corollary 1.3. $\square$

Corollary 2.2. $\psi(t) \to 0$ *exponentially as* $t \to \pm\infty$.

From the corollary, in (2),

$$\lim_{T\to\infty} \psi(-T)\frac{\partial x_-}{\partial \mu}(-T,0) = 0 \cdot \frac{\partial p_-}{\partial \mu}(0) = 0.$$

Therefore

$$\frac{d\xi_-}{d\mu}(0) = \int_{-\infty}^{0} \psi(s)\frac{\partial f}{\partial \mu}(x_*(s),0)ds.$$

Similarly,

$$\frac{d\xi_+}{d\mu}(0) = \int_{\infty}^{0} \psi(s)\frac{\partial f}{\partial \mu}(x_*(s),0)ds.$$

Therefore

$$S'(0) = \xi_-'(0) - \xi_+'(0) = \int_{-\infty}^{\infty} \psi(s)\frac{\partial f}{\partial \mu}(x_*(s),0)ds.$$

The integral is called a Melnikov integral.