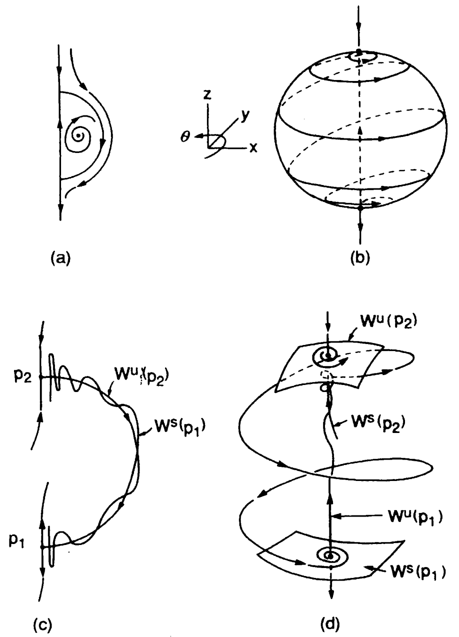

FIGURE 3.1.43.

in Chapter 4 that chaotic dynamics may result.

Step 5: Analysis of Global Bifurcations. As we have mentioned, this will be completed in Section 4.9 after we have developed the necessary theoretical tools.

Step 6: Effects of the Higher Order Terms in the Normal Form. In Cases I and IVa,b the method of averaging essentially enables us to conclude that the higher order terms do not qualitatively change the dynamics. Thus we have found a versal deformation. The details of proving this, however, are left to the exercises.

The remaining cases are more difficult and, ultimately, we will argue that versal deformations may not exist in some circumstances.

Before leaving this section we want to make some final remarks.

Remark 1. This analysis reemphasizes the power of the method of normal forms. As we will see throughout the remainder of this book, vector fields having phase spaces of dimension three or more can exhibit very complicated dynamics. In our case the method of normal forms utilized the structure of the vector field to naturally "separate" the variables. This

enabled us to "get our foot in the door" by using powerful phase plane techniques.

Remark 2. From the double-zero eigenvalue and now this case, a lesson to be learned is that Poincaré-Andronov-Hopf bifurcations always cause us trouble in the sense of how they relate to global bifurcations and/or how they are affected by the consideration of the higher order terms of the normal form.

3.2 Bifurcations of Fixed Points of Maps

The theory for bifurcations of fixed points of maps is very similar to the theory for vector fields. Therefore, we will not include as much detail but merely highlight the differences when they occur.

Consider a p-parameter family of maps of $\mathbb{R}^n$ into $\mathbb{R}^n$

$$y \mapsto g(y, \lambda), \qquad y \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^p \tag{3.2.1}$$

where g is $\mathbf{C}^r$ (with r to be specified later, usually $r \geq 5$ is sufficient) on some sufficiently large open set in $\mathbb{R}^n \times \mathbb{R}^p$. Suppose (3.2.1) has a fixed point at $(y, \lambda) = (y_0, \lambda_0)$, i.e.,

$$g(y_0, \lambda_0) = y_0. \tag{3.2.2}$$

Then, just as in the case for vector fields, two questions naturally arise.

1. Is the fixed point stable or unstable?
2. How is the stability or instability affected as λ is varied?

As in the case for vector fields, an examination of the associated linearized map is the first place to start in order to answer these questions. The associated linearized map is given by

$$\xi \mapsto D_y g(y_0, \lambda_0)\xi, \qquad \xi \in \mathbb{R}^n, \tag{3.2.3}$$

and, from Sections 1.1A and 1.1C, we know that if the fixed point is hyperbolic (i.e., none of the eigenvalues of $D_y g(y_0, \lambda_0)$ have unit modulus), then stability (resp. instability) in the linear approximation implies stability (resp. instability) of the fixed point of the nonlinear map. Moreover, using an implicit function theorem argument exactly like that given at the beginning of Section 3.1, it can be shown that, in a sufficiently small neighborhood of (y_0, λ_0), for each λ there is a unique fixed point having the same stability type as (y_0, λ_0). Thus, hyperbolic fixed points are locally dynamically dull!

The fun begins when we consider Questions 1 and 2 above in the situation when the fixed point is *not hyperbolic*. Just as in the case for vector fields, the linear approximation cannot be used to determine stability, and varying λ can result in the creation of new orbits (i.e., bifurcation). The simplest ways in which a fixed point of a map can be nonhyperbolic are the following.

1. $D_y g(y_0, \lambda_0)$ has a single eigenvalue equal to 1 with the remaining $n-1$ eigenvalues having moduli not equal to 1.

2. $D_y g(y_0, \lambda_0)$ has a single eigenvalue equal to -1 with the remaining $n-1$ eigenvalues having moduli not equal to 1.

3. $D_y g(y_0, \lambda_0)$ has two complex conjugate eigenvalues having modulus 1 (which are *not* one of the first four roots of unity) with the remaining $n-2$ eigenvalues having moduli not equal to 1.

Using the center manifold theory, the analysis of the above situations can be reduced to the analysis of a p-parameter family of one-, one-, and two-dimensional maps, respectively. We begin with the first case.

3.2A An Eigenvalue of 1

In this case, the study of the orbit structure near the fixed point can be reduced to the study of a parametrized family of maps on the one-dimensional center manifold. We suppose that the map on the center manifold is given by

$$x \mapsto f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{3.2.4}$$

where, for now, we will consider only one parameter (if there is more than one parameter in the problem, we will consider all but one as fixed constants). In making the reduction to the center manifold, the fixed point $(y_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}^p$ has been transformed to the origin in $\mathbb{R}^1 \times \mathbb{R}^1$ (cf. Section 2.1A) so that we have

$$f(0,0) = 0, \tag{3.2.5}$$

$$\frac{\partial f}{\partial x}(0,0) = 1. \tag{3.2.6}$$

i) The Saddle-Node Bifurcation

Consider the map

$$x \mapsto f(x, \mu) = x + \mu \mp x^2, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.2.7}$$

It is easy to verify that $(x, \mu) = (0,0)$ is a nonhyperbolic fixed point of (3.2.7) with eigenvalue 1, i.e.,

$$f(0,0) = 0, \tag{3.2.8}$$

$$\frac{\partial f}{\partial x}(0,0) = 1. \tag{3.2.9}$$

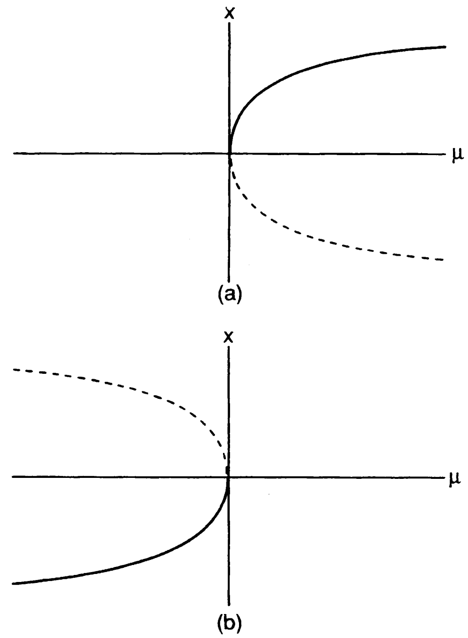

FIGURE 3.2.1. a) $f(x,\mu) = x + \mu - x^2$; b) $f(x,\mu) = x + \mu + x^2$.

We are interested in the nature of the fixed points for (3.2.7) near $(x, \mu) = (0,0)$. Since (3.2.7) is so simple, we can solve for the fixed points directly as follows

$$f(x, \mu) - x = \mu \mp x^2 = 0. \tag{3.2.10}$$

We show the two curves of fixed points in Figure 3.2.1 and leave it as an exercise for the reader to verify the stability types of the different branches of fixed points shown in this figure. We refer to the bifurcation occuring at $(x, \mu) = (0,0)$ as a *saddle-node* bifurcation.

In analogy with the situation for vector fields (see Section 3.1A) we want to find general conditions (in terms of derivatives evaluated at the bifurcation point) under which a map will undergo a saddle-node bifurcation, i.e.,

> the map possesses a unique curve of fixed points in the $x - \mu$ plane passing through the bifurcation point which locally lies on one side of $\mu = 0$.

We proceed using the implicit function theorem exactly as in the case for vector fields.

Consider a general one-parameter family of one-dimensional maps

$$x \mapsto f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{3.2.11}$$

with

$$f(0,0) = 0, \tag{3.2.12}$$

$$\frac{\partial f}{\partial x}(0,0) = 1. \tag{3.2.13}$$

The fixed points of (3.2.11) are given by

$$f(x,\mu) - x \equiv h(x,\mu) = 0. \tag{3.2.14}$$

We seek conditions under which (3.2.14) defines a curve in the $x-\mu$ plane with the properties described above. By the implicit function theorem,

$$\frac{\partial h}{\partial \mu}(0,0) = \frac{\partial f}{\partial \mu}(0,0) \neq 0 \tag{3.2.15}$$

implies that a single curve of fixed points passes through $(x,\mu) = (0,0)$; moreover, for x sufficiently small, this curve of fixed points can be represented as a graph over the x variables, i.e., there exists a unique $\mathbf{C}^r$ function, $\mu(x)$, x sufficiently small, such that

$$h(x, \mu(x)) \equiv f(x, \mu(x)) - x = 0. \tag{3.2.16}$$

Now we simply require that

$$\frac{d\mu}{dx}(0) = 0, \tag{3.2.17}$$

$$\frac{d^2\mu}{dx^2}(0) \neq 0. \tag{3.2.18}$$

As was the case for vector fields (Section 3.1A), we obtain (3.2.17) and (3.2.18) in terms of derivatives of the map at the bifurcation point by implicitly differentiating (3.2.16). Following (3.1.40) and (3.1.43), we obtain

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial h}{\partial x}(0,0)}{\frac{\partial h}{\partial \mu}(0,0)} = -\frac{\left(\frac{\partial f}{\partial x}(0,0) - 1\right)}{\frac{\partial f}{\partial \mu}(0,0)} = 0, \tag{3.2.19}$$

$$\frac{d^2\mu}{dx^2}(0) = \frac{-\frac{\partial^2 h}{\partial x^2}(0,0)}{\frac{\partial h}{\partial \mu}(0,0)} = \frac{-\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)}. \tag{3.2.20}$$

To summarize, a general one-parameter family of $\mathbf{C}^r$ $(r \geq 2)$ one-dimensional maps

$$x \mapsto f(x,\mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1$$

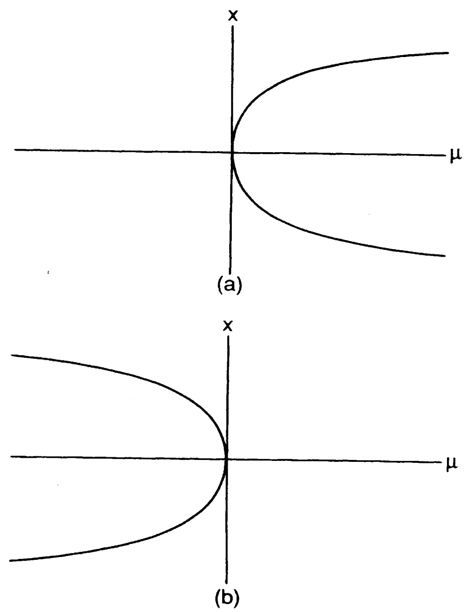

FIGURE 3.2.2. a) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)\right) > 0$; b) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)\right) < 0$.

undergoes a *saddle-node* bifurcation at $(x,\mu) = (0,0)$ if

$$\left.\begin{aligned} f(0,0) &= 0 \\ \tfrac{\partial f}{\partial \mu}(0,0) &= 1 \end{aligned}\right\} \quad \text{nonhyperbolic fixed point} \tag{3.2.21}$$

with

$$\frac{\partial f}{\partial \mu}(0,0) \neq 0, \tag{3.2.22}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0. \tag{3.2.23}$$

Moreover, the sign of (3.2.20) tells us on which side of $\mu = 0$ the curve of fixed points is located; we show the two cases in Figure 3.2.2 and leave it as an exercise for the reader to compute the possible stability types of the branches of fixed points shown in the figure (see Exercise 3.5). Thus, (3.2.7) can be viewed as a normal form for the saddle-node bifurcation of maps. Notice that, with the exception of the condition $\frac{\partial f}{\partial x}(0,0) = 1$, the conditions for a one-parameter family of one-dimensional maps to undergo a saddle-node bifurcation in terms of derivatives of the map at the bifurcation point are exactly the same as those for vector fields (cf. (3.1.46), (3.1.47) and (3.1.48)). The reader should consider the implications of this.

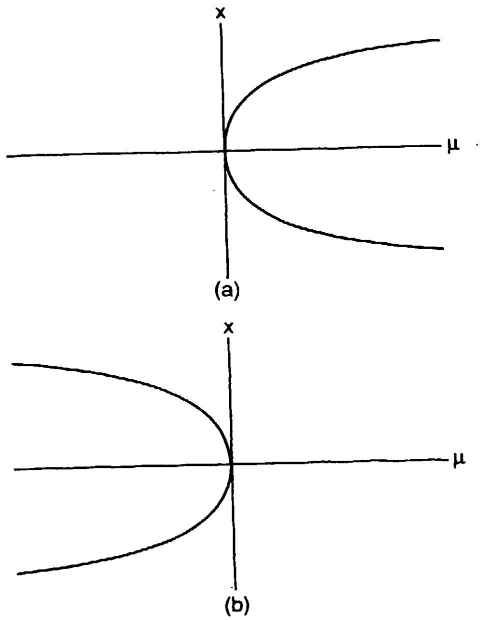

FIGURE 3.2.8. a) $\left(-\frac{\partial^3 f}{\partial x^3}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) > 0$; b) $\left(-\frac{\partial^3 f}{\partial x^3}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) < 0$.

undergoes a pitchfork bifurcation at $(x, \mu) = (0, 0)$ if

$$\frac{\partial f}{\partial \mu}(0,0) = 0, \tag{3.2.75}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0, \tag{3.2.76}$$

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0, \tag{3.2.77}$$

$$\frac{\partial^3 f}{\partial x^3}(0,0) \neq 0. \tag{3.2.78}$$

Moreover, the sign of (3.2.71) tells us on which side of $\mu = 0$ that one of the curves of fixed points lies. We illustrate both cases in Figure 3.2.8 and leave it as an exercise (see Exercise 3.7) for the reader to compute the possible stability types of the different branches shown in Figure 3.2.8. Thus, we can view (3.2.50) as a normal form for the pitchfork bifurcation.

We end our discussion of the pitchfork bifurcation by graphically showing the bifurcation for

$$x \mapsto x + \mu x - x^3$$

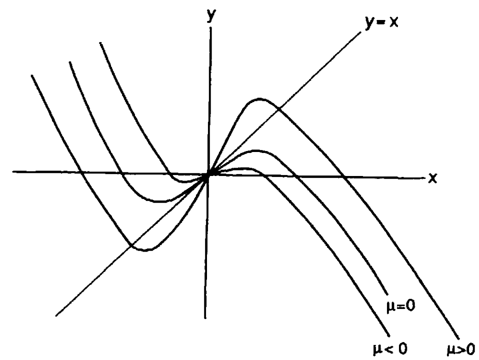

FIGURE 3.2.9.

in Figure 3.2.9 in the manner discussed at the end of Section 3.2A,i).

3.2B An Eigenvalue of -1

Suppose that our one-parameter family of $\mathbf{C}^r$ ($r \geq 3$) one-dimensional maps has a nonhyperbolic fixed point, and the eigenvalue associated with the linearization of the map about the fixed point is -1 rather than 1. Up to this point the bifurcations of one-parameter families of one-dimensional maps have been very much the same as the analogous cases for vector fields. However, the case of an eigenvalue equal to -1 is fundamentally different and does not have an analog with *one-dimensional* vector field dynamics. We begin by studying a specific example.

i) Example

Consider the following one-parameter family of one-dimensional maps

$$x \mapsto f(x, \mu) = -x - \mu x + x^3, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.2.79}$$

It is easy to verify that (3.2.79) has a nonhyperbolic fixed point at $(x, \mu) = (0, 0)$ with eigenvalue -1, i.e.,

$$f(0,0) = 0, \tag{3.2.80}$$

$$\frac{\partial f}{\partial x}(0,0) = -1. \tag{3.2.81}$$

The fixed points of (3.2.79) can be calculated directly and are given by

$$f(x, \mu) - x = x(x^2 - (2 + \mu)) = 0. \tag{3.2.82}$$

Thus, (3.2.79) has two curves of fixed points,

$$x = 0 \tag{3.2.83}$$

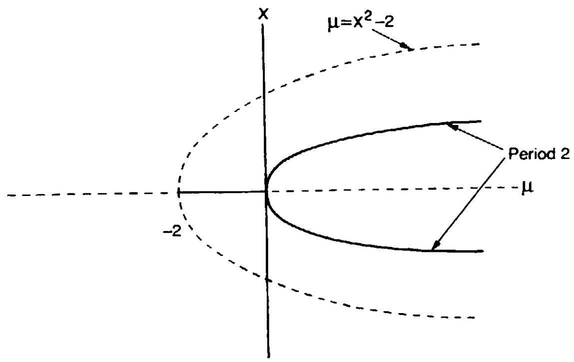

FIGURE 3.2.10.

and

$$x^2 = 2 + \mu, \tag{3.2.84}$$

but only (3.2.83) passes through the bifurcation point $(x, \mu) = (0, 0)$. In Figure 3.2.10 we illustrate the two curves of fixed points and leave it as an exercise for the reader to verify the stability types for the different curves of fixed points shown in the figure. In particular we have

$$x = 0 \text{ is } \begin{cases} \text{unstable for } \mu \le -2, \\ \text{stable for } -2 < \mu < 0, \\ \text{unstable for } \mu > 0, \end{cases} \tag{3.2.85}$$

and

$$x^2 = 2 + \mu \text{ is } \begin{cases} \text{unstable for } \mu \ge -2, \\ \text{does not exist for } \mu < -2. \end{cases} \tag{3.2.86}$$

From (3.2.85) and (3.2.86) we can immediately see there is a problem, namely, that for $\mu > 0$, the map has exactly three fixed points and all are unstable. (Note: this situation could not occur for one-dimensional vector fields!) A way out of this difficulty would be provided if stable periodic orbits bifurcated from $(x, \mu) = (0, 0)$. We will see that this is indeed the case.

Consider the *second iterate* of (3.2.79), i.e.,

$$x \mapsto f^2(x, \mu) = x + \mu(2 + \mu)x - 2x^3 + \mathcal{O}(4). \tag{3.2.87}$$

It is easy to verify that (3.2.87) has a nonhyperbolic fixed point at $(x, \mu) = (0, 0)$ having an eigenvalue of 1, i.e.,

$$f^2(0, 0) = 0, \tag{3.2.88}$$

$$\frac{\partial f^2}{\partial x}(0, 0) = 1. \tag{3.2.89}$$

Moreover,

$$\frac{\partial f^2}{\partial \mu}(0, 0) = 0, \tag{3.2.90}$$

$$\frac{\partial^2 f^2}{\partial x \partial \mu}(0, 0) = 2, \tag{3.2.91}$$

$$\frac{\partial^2 f^2}{\partial x^2}(0, 0) = 0, \tag{3.2.92}$$

$$\frac{\partial^3 f^2}{\partial x^3}(0, 0) = -12. \tag{3.2.93}$$

Hence, from (3.2.75), (3.2.76), (3.2.77), and (3.2.78), (3.2.90), (3.2.91), (3.2.92), and (3.2.93) imply that the second iterate of (3.2.79) undergoes a pitchfork bifurcation at $(x, \mu) = (0, 0)$. Since the new fixed points of $f^2(x, \mu)$ are not fixed points of $f(x, \mu)$, they must be period two points of $f(x, \mu)$. Hence, $f(x, \mu)$ is said to have undergone a *period-doubling bifurcation* at $(x, \mu) = (0, 0)$.

ii) The Period-Doubling Bifurcation

Consider a one-parameter family of $\mathbf{C}^r$ $(r \ge 3)$ one-dimensional maps

$$x \mapsto f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.2.94}$$

We seek conditions for (3.2.94) to undergo a period-doubling bifurcation. The previous example will be our guide. It should be clear from the example that conditions sufficient for (3.2.94) to undergo a period-doubling bifurcation are for the map to have a nonhyperbolic fixed point with eigenvalue −1 and for the second iterate of the map to undergo a pitchfork bifurcation at the same nonhyperbolic fixed point. To summarize, using (3.2.73), (3.2.74), (3.2.75), (3.2.76), (3.2.77), and (3.2.78), it is sufficient for (3.2.94) to satisfy

$$f(0, 0) = 0, \tag{3.2.95}$$

$$\frac{\partial f}{\partial x}(0, 0) = -1, \tag{3.2.96}$$

$$\frac{\partial f^2}{\partial \mu}(0, 0) = 0, \tag{3.2.97}$$

$$\frac{\partial^2 f^2}{\partial x^2}(0, 0) = 0, \tag{3.2.98}$$

$$\frac{\partial^2 f^2}{\partial x \partial \mu}(0, 0) \ne 0, \tag{3.2.99}$$

$$\frac{\partial^3 f^2}{\partial x^3}(0, 0) \ne 0. \tag{3.2.100}$$

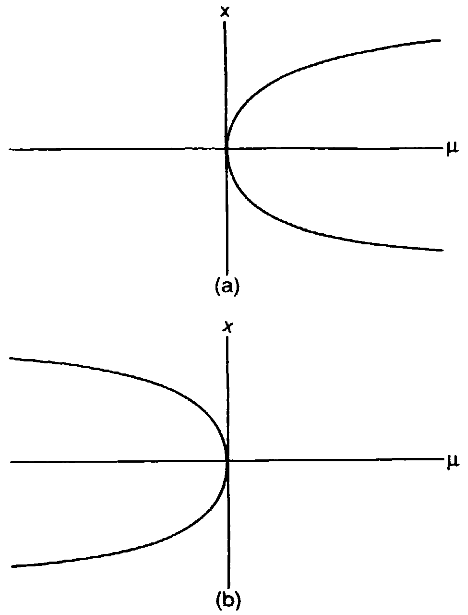

FIGURE 3.2.11. a) $(-\frac{\partial^3 f^2}{\partial x^3}(0,0)/\frac{\partial^2 f^2}{\partial x \partial \mu}(0,0)) > 0$; b) $(-\frac{\partial^3 f^2}{\partial x^3}(0,0)/\frac{\partial^2 f^2}{\partial x \partial \mu}(0,0)) < 0$.

Moreover, the sign of $\left(-\frac{\partial^3 f^2(0,0)}{\partial x^3} \Big/ \frac{\partial^2 f^2(0,0)}{\partial x \partial \mu}\right)$ tells us on which side of $\mu = 0$ the period two points lie. We show both cases in Figure 3.2.11 and leave it as an exercise for the reader to compute the possible stability types for the different curves of fixed points shown in the figure; see Exercise 3.8.

Finally, we demonstrate graphically the period-doubling bifurcation for

$$x \mapsto -x - \mu x + x^3 \equiv f(x, \mu)$$

and the associated pitchfork bifurcation for $f^2(x, \mu)$ in the graphical manner discribed at the end of Section 3.2A, i) in Figure 3.2.12.

3.2c A Pair of Eigenvalues of Modulus 1: The Naimark-Sacker Bifurcation

This section describes the map analog of the Poincaré-Andronov-Hopf bifurcation for vector fields but with some very different twists. Although this bifurcation often goes by the name of "Hopf bifurcation for maps," this is misleading because the bifurcation theorem was first proved independently by Naimark [1959] and Sacker [1965]. Consequently, we will use the term

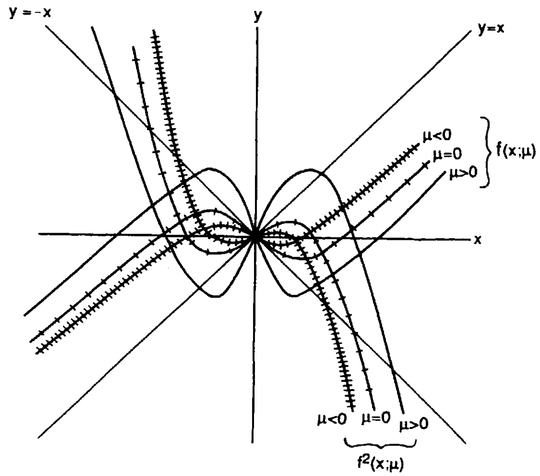

FIGURE 3.2.12.

"Naimark-Sacker bifurcation."

We know that in this situation the study of the dynamics of (3.2.1) near the fixed point $(y_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}^p$ can be reduced to the study of (3.2.1) restricted to a p-parameter family of two-dimensional center manifolds. We assume that the reduced map has been calculated and is given by

$$x \mapsto f(x, \mu), \qquad x \in \mathbb{R}^2, \quad \mu \in \mathbb{R}^1, \tag{3.2.101}$$

where we take $p = 1$. If there is more than one parameter, we consider all but one as fixed and denote it as μ. In restricting the map to the center manifold, some preliminary transformations have been made so that the fixed point of (3.2.101) is given by $(x, \mu) = (0, 0)$, i.e., we have

$$f(0, 0) = 0, \tag{3.2.102}$$

with the matrix

$$D_x f(0, 0) \tag{3.2.103}$$

having two complex conjugate eigenvalues, denoted $\lambda(0), \bar{\lambda}(0)$, with

$$|\lambda(0)| = 1. \tag{3.2.104}$$

We will also require that

$$\lambda^n(0) \neq 1, \qquad n = 1, 2, 3, 4. \tag{3.2.105}$$

(Note: if $\lambda(0)$ satisfies (3.2.105), then so does $\lambda(\bar{0})$, and vice versa.)