TWO THEOREMS ABOUT LINEAR OPERATORS

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Theorem 1. *If X is a normed vector space and Y is a Banach space, then $L(X,Y)$ is a Banach space.*

Recall that $L(X,Y)$ is the vector space of bounded linear maps from X to Y, with the norm

$$\|A\| = \sup_{x\,:\,|x|=1} |Ax|.$$

Proof. Let A_n be a Cauchy sequence in $L(x,y)$. "Define" a function $A : X \to Y$ as follows: for each $x \in X$, $Ax = \lim_{n\to\infty} A_n x$. This makes sense provided $\lim_{n\to\infty} A_n x$ exists. Since $A_n x$ is a sequence in the Banach space Y, the limit exists provided the sequence is Cauchy. There is no problem when $x = 0$ since $A_n x = 0$ for every n. So let $x \in X$ with $x \neq 0$. Let $\epsilon > 0$. Choose N such that for n, $m \geq N$, $\|A_n - A_m\| < \frac{\epsilon}{|x|}$. We can do this because the sequence A_n is Cauchy. Then for n, $m \geq N$,

$$|(A_n - A_m)x| \leq \|A_n - A_m\||x| < \frac{\epsilon}{|x|}|x| = \epsilon.$$

Therefore the sequence $A_n x$ is Cauchy, so we have defined Ax.

We must show:

(1) A is linear.
(2) A is bounded.
(3) $A_n \to A$, i.e., $\|A_n - A\| \to 0$.

(1) A is linear:

$$A(x_1 + x_2) = \lim_{n\to\infty} A_n(x_1 + x_2) = \lim_{n\to\infty} (A_n x_1 + A_n x_2) = \lim_{n\to\infty} A_n x_1 + \lim_{n\to\infty} A_n x_2 = Ax_1 + Ax_2.$$

The argument that $A(ax) = aAx$ is similar.

(2) A is bounded:

The sequence A_n is Cauchy, so it is bounded, in the sense that there is a number $C > 0$ such that $\|A_n\| \leq C$ for all n. (This is a general fact about Cauchy sequences in a normed vector space.) Therefore, for any n and any $x \in X$, $|A_n x| \leq C|x|$. Hence, for any fixed $x \in X$,

$$|Ax| = \lim_{n\to\infty} |A_n x| \leq C|x|.$$

(This uses the general fact that in a normed vector space, if $y_n \to y$ and $|y_n| \leq c$ for all n, then $|y| \leq c$.) Therefore A is bounded.

(3) $\|A_n - A\| \to 0$:

Let $\epsilon > 0$. Since the sequence A_n is Cauchy, there is a number N such that for n, $m \geq N$, $\|A_n - A_m\| < \frac{\epsilon}{2}$. Let $n \geq N$. We claim that $\|A_n - A\| \leq \epsilon$, which proves the result. To see

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this, let $x \in X$ with $|x| = 1$. We just need to show that that $|(A_n - A)x| \le \epsilon$. Choose $m \ge N$ such that $|A_m x - Ax| < \frac{\epsilon}{2}$. (We can do this because of the way we defined Ax above.) Then

$$\begin{aligned}|(A_n - A)x| &= |A_n x - Ax| \le |A_n x - A_m x| + |A_m x - Ax| = |(A_n - A_m)x| + |A_m x - Ax| \\ &\le \|A_n - A_m\||x| + |A_m x - Ax| = \|A_n - A_m\| + |A_m x - Ax| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

Theorem 2. *Let X be a normed vector space, let Z be a subspace of X, let Y be a Banach space, let $A : Z \to Y$ be a bounded linear map, and let $\bar{Z}$ denote the closure of Z in X. Then there is a unique bounded linear map $B : \bar{Z} \to Y$ such that $B|Z = A$. Moreover, $\|B\| = \|A\|$.*

Proof. If $x \in Z$, define Bx to be Ax (of course). If $x \in \bar{Z} \setminus Z$, then there is a sequence x_n in Z such that $x_n \to x$. We claim that Ax_n is a Cauchy sequence in Y. To see this, let $\epsilon > 0$. Since x_n converges, it is Cauchy, so there is a number N such that for n, $m \ge N$, $|x_n - x_m| < \frac{\epsilon}{\|A\|}$. Then for n, $m \ge N$,

$$|Ax_n - Ax_m| = |A(x_n - x_m)| \le \|A\||x_n - x_m| < \|A\|\frac{\epsilon}{\|A\|} = \epsilon.$$

Since Ax_n is a Cauchy sequence in Y, it converges to some $y \in Y$. "Define" $Bx = y$.

We must show:

(1) B is well-defined, i.e., the definition of Bx is independent of which sequence $x_n \to x$ we use to define it.
(2) B is linear.
(3) $\|B\| = \|A\|$.
(4) Items (1)-(3) imply that B is a bounded linear extension of A to $\bar{Z}$. For the uniqueness, we should also explain why there is no other bounded linear extension of A to $\bar{Z}$.

(1) Let x'_n be another sequence in Z that approaches $x \in \bar{Z} \setminus Z$. By the argument above there exists $y' \in Y$ such that $Ax'_n \to y'$. We must show that $y = y'$.

For any n,

$$|y - y'| \le |y - Ax_n| + |Ax_n - Ax'_n| + |Ax'_n - y'| \le |y - Ax_n| + \|A\||x_n - x'_n| + |Ax'_n - y'|.$$

Each of the three summands on the right approaches 0 as $n \to \infty$, so $|y - y'| = 0$, so $y = y'$.

(2) Let x, $x' \in \bar{Z}$. Choose sequences x_n and x'_n in Z such that $x_n \to x$ and $x'_n \to x'$. Then $x_n + x'_n \to x + x'$, so

$$B(x + x') = \lim A(x_n + x'_n) = \lim(Ax_n + Ax'_n) = \lim Ax_n + \lim Ax'_n = Bx + Bx'.$$

The proof that $B(ax) = aBx$ is similar.

(3) For $x \in Z$ we have $|Bx| = |Ax| \le \|A\||x|$. For $x \in \bar{Z} \setminus Z$, let x_n be a sequence in Z such that $x_n \to x$. Then

$$|Bx| = |\lim Ax_n| = \lim |Ax_n| \le \lim \|A\||x_n| \le \|A\| \lim |x_n| = \|A\||x|.$$

It follows that $\|B\| \le \|A\|$. But just considering $x \in Z$ shows that $\|B\| \ge \|A\|$. Therefore $\|B\| = \|A\|$.

(4) If B' is any bounded extension of A to $\bar{Z}$, then B' is continuous. Let $x \in \bar{Z} \setminus Z$,and let x_n be a sequence in Z such that $x_n \to x$. Then

$$B'x = \lim B'x_n = \lim Ax_n = Bx.$$

Therefore $B' = B$.

$\square$