MA 732 Homework 6

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1. The differential equation

$$\dot{x} = -y + x(x^2 + y^2 - 1)^2,$$
$$\dot{y} = x + y(x^2 + y^2 - 1)^2$$

has a closed orbit, namely the circle $x^2 + y^2 = 1$. Show that this closed orbit is not hyperbolic. In other words, let $P(x)$ be the Poincaré map on the positive x-axis, so that $P(1) = 1$. Show that $P'(1) = 1$. Suggestion: use polar coordinates. Don't try to compute the Poincaré map and differentiate it. Instead solve the variational equation along the closed orbit.

2. Consider the differential equation

$$\ddot{x} + \mu(x^2 + \dot{x}^2 - 1)\dot{x} + x = 0.$$

For $\mu = 0$, all solutions are periodic. Which periodic solutions persist for small μ? (Write the differential equation as a system, then use Theorem 3 in the handout "Persistence of periodic solutions in a one-parameter family." For this problem, both $T(\xi, 0)$ and $\psi(t, \xi)$ are independent of ξ.)

3. Suppose the differential equation $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, has an equilibrium x_0. Let $T > 0$. Prove: if $Df(x_0)$ has no eigenvalue of the form $2m\pi i/T$, m an integer, then there are no closed orbits of period T near x_0 other than $x \equiv x_0$. Suggestion: Consider the following differential equation on $\mathbb{R}^n \times \mathbb{R}$:

$$\dot{x} = f(x),$$
$$\dot{\tau} = 1.$$

Define a "Poincaré map" P from the "plane" $\tau = 0$ to the "plane" $\tau = T$ by following solutions. Then $P(x_0) = x_0$. Show that $DP(x_0)$ does not have 1 as an eigenvalue. Then use the Inverse Function Theorem to argue that for the displacement function $d(x) = P(x) - x$, there are no solutions to the equation $d(x) = 0$ near x_0 other than $x = x_0$.

4. Consider the periodically forced Van der Pol equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = \epsilon \sin \Omega t.$$

As we did in class, let

$$\Omega(\epsilon) = \frac{m}{n} - \frac{k}{2\pi}\left(\frac{m}{n}\right)^2 \epsilon + \mathcal{O}(\epsilon^2),$$

so that ϵ is the only parameter in our equation. Using the formula we derived in class, compute the first-order (in ϵ) approximation to the Poincaré when $m/n \neq 1$. In other words, use our formula

$$P^m(y, \epsilon) = y + \epsilon\Delta(y, \epsilon)$$

and our formula for $\Delta(y, 0)$ to compute the map

$$P^m_{\text{approx}}(y, \epsilon) = y + \epsilon\Delta(y, 0).$$

I believe you will get

$$P^m_{\text{approx}}((y_1, y_2), \epsilon) = \begin{pmatrix} y_1 + \epsilon\left(kmy_2 + n\pi y_1 - \frac{n\pi}{4} y_1(y_1^2 + y_2^2)\right) \\ y_2 + \epsilon\left(-kmy_1 + n\pi y_2 - \frac{n\pi}{4} y_2(y_1^2 + y_2^2)\right) \end{pmatrix}.$$

Then convert to polar coordinates (remember, this is a map, not a differential equation) to get

$$P^m_{\text{approx}}((r, \theta), \epsilon) = \begin{pmatrix} r + \epsilon n\pi r\left(1 - \frac{r^2}{4}\right) \\ \theta - \epsilon km \end{pmatrix}.$$

Notice that P^m_{approx} has the invariant circle $r = 2$ (if you start on it, you stay on it). On this circle, θ changes by ϵmk with each iteration. This calculation suggests, but does not prove, that for small ϵ, $P^m(y, \epsilon)$ has an invariant circle near $r = 2$. We have not studied how to prove that a map has an invariant circle.