MA 732 Homework 5

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1. Consider the system

$$\dot{x} = x - y - x^2(x+3y) - y^2(x+y),$$
$$\dot{y} = x + y + x^2(x-y) - y^2(x+y).$$

   (a) Check that the origin is a repelling focus.

   (b) Use polar coordinates to find an annulus that is guaranteed, by the Poincaré-Bendixson Theorem, to have a closed orbit.

2. Consider the differential equation $\dot{x} = y + x - \frac{x^3}{3}$, $\dot{y} = -x$.

   (a) Draw the nullclines and the vector field on the nullclines. Use this information to sketch the approximate direction of the vector field in the open regions determined by the nullclines.

   (b) From the above information alone, does it seem possible that there is a closed orbit that lies between the lines $x = -1$ and $x = 1$?

   (c) Use Bendixson's Criterion to show that there is no closed orbit that lies between the lines $x = -1$ and $x = 1$.

   (d) Show that our equation has the symmetry $R(x, y) = (-x, -y)$. (Thinking of our system as $\dot{z} = f(z)$, you want to show that $f(R(z)) = DR(z)f(z)$.) This symmetry implies that if $(x(t), y(t))$ is a solution of our differential equation, then $(\tilde{x}(t), \tilde{y}(t)) = (-x(t), -y(t))$ is also a solution. (You could also check this directly.)

   (e) Let $(x(t), y(t))$, $0 \leq t \leq T$ be a solution with $(x(0), y(0)) = (0, a)$, $a > 0$; $(x(T), y(T)) = (0, -b)$, $b > 0$; and $x(t) > 0$ for $0 < t < T$. Show: if $a \neq b$, then the curve $(x(t), y(t))$, $0 \leq t \leq T$, is not part of a closed orbit. Suggestion: Just treat one of the cases $a < b$ or $a > b$. Draw the curves $(x(t), y(t))$, $0 \leq t \leq T$, and $(\tilde{x}(t), \tilde{y}(t))$, $0 \leq t \leq T$ (see part (d)), and study your picture.

   (f) Use the work you have done to show that any closed orbit must cross both the line $x = -1$ and the line $x = 1$.

3. Consider the differential equation

$$x'' + \epsilon(x^2 - 1)x' + x = 0, \tag{1}$$

a version of van der Pol's equation. For $\epsilon = 0$ all solutions have period 2π. In lecture we used the Poincaré-Lindstedt method to compute, for small $\epsilon > 0$, a periodic solution with amplitude near 2. In this problem we will use an ordinary perturbation calculation to compute the Poincaré map (not just its fixed point).

Written as a system, our equation becomes

$$x' = y,$$
$$y' = -x - \epsilon(x^2 - 1)y.$$

We will compute the Poincaré map on the positive x-axis. For $\epsilon = 0$, the solution that starts at $(b, 0)$ with $b > 0$ returns to the point $(b, 0)$ after time 2π. More generally, for a given small $\epsilon > 0$, the solution that starts at $(b, 0)$ returns to a point $(P(b, \epsilon), 0)$ on the positive x-axis after time $\tau(b, \epsilon)$, with $P(b, 0) = b$ and $\tau(b, 0) = 2\pi$. We wish to compute solutions that start at $(b, 0)$, namely

$$\phi(t, (b, 0), \epsilon) = (x(t, b, \epsilon), y(t, b, \epsilon)).$$

The time $\tau(b, \epsilon)$ satisfies the equation $y(\tau(b, \epsilon), b, \epsilon) = 0$ with $\tau(b, 0) = 2\pi$. Then

$$P(b, \epsilon) = x(\tau(b, \epsilon), b, \epsilon).$$

Actually, we only need to calculate $x(t, b, \epsilon)$. The time $\tau(b, \epsilon)$ satisfies the equation $x'(\tau(b, \epsilon), b, \epsilon) = 0$ with $\tau(b, 0) = 2\pi$.

To compute the expansion of $x(t, b, \epsilon)$, we can ignore τ and write:

$$x(t, b, \epsilon) = x_0(t, b) + \epsilon x_1(t, b) + \epsilon^2 x_2(t, b) + \dots, \quad \tau(b, \epsilon) = 2\pi + \epsilon\tau_1(b) + \epsilon^2\tau_2(b) + \dots.$$

Substituting the first equation into (1), we have

$$x_0'' + \epsilon x_1'' + \epsilon^2 x_2'' + \dots + \epsilon\Big((x_0 + \epsilon x_1 + \dots)^2 - 1\Big)\Big(x_0' + \epsilon x_1' + \dots\Big) + x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots = 0. \quad (2)$$

Initial conditions are $x(0, b, \epsilon) = b$, $x'(0, b, \epsilon) = y(0, b, \epsilon) = 0$. Therefore

$$x_0(b, 0) = b,\ x_1(b, 0) = 0,\ x_2(b, 0) = 0,\ \dots,$$
$$x_0'(b, 0) = 0,\ x_1'(b, 0) = 0,\ x_2'(b, 0) = 0,\ \dots. \quad (3)$$

Equating corresponding terms in (2), using (3), and suppressing the parameter b to simplify the notation, we have at the ϵ^0 level:

$$x_0'' + x_0 = 0, \quad x_0(0) = b, \quad x_0'(0) = 0.$$

The solution is $x_0 = b\cos t$.

If you get stuck on the following problems, look at Example 3 in Section 6.5.3 of the text.

(a) Write the corresponding equation and initial conditions at the ϵ^1 level. Solve. You now have $x(t, b, \epsilon)$ to first order in ϵ.

(b) Calculate $x'(t, b, \epsilon)$ to first order in ϵ by differentiating the previous expression with respect to t.

(c) Write the equation $x'(t, b, \epsilon) = 0$ to first order in ϵ. Substitute $t = \tau_0 + \epsilon\tau_1 + \dots$ and solve for τ_0 and τ_1.

(d) Calculate $P(b, \epsilon)$ to first order in ϵ by substituting your formula for t into your formula for $x(t, b, \epsilon)$ and gathering terms.

(e) There is a fixed point of the Poincare map of the form $b(\epsilon) = b_0 + \epsilon b_1 + \dots$ with $b_0 > 0$. Calculate b_0 from your formula for $P(b, \epsilon)$. (You want to solve the equation $P(b, \epsilon) - b = 0$.)

(f) This fixed point is hyperbolic for $\epsilon > 0$. To see this, calculate $\frac{\partial P}{\partial b}(b(\epsilon), \epsilon)$ to lowest order in ϵ.

4. Use center manifold reduction to describe the phase portrait near the origin for the system

$$\begin{aligned}\dot{x} &= -y + xy,\\ \dot{y} &= x + yz,\\ \dot{z} &= -z - x^2 - y^2 + z^2.\end{aligned}$$

Suggestion: The eigenvalues are $\pm i$ and -1. The center subspace is the xy-plane. The center manifold is therefore

$$z = h(x, y) = 0 + 0 \cdot x + 0 \cdot y + Ax^2 + Bxy + Cy^2 + \dots.$$

Find A, B, and C, and use them to find the differential equation on the center manifold to third order. Use polar coordinates to see what its flow is. This should enable you to describe the phase portrait near the origin, which is determined by the flow on the center manifold and the fact that there is a stable manifold but no unstable manifold.

5. Let $\dot{x} = f(x, \mu) = xa(x, \mu)$, where $a(x, \mu)$ is C^1, $a(0, 0) = 0$, $a_x(0, 0) = H \neq 0$, and $a_\mu(0, 0) = I \neq 0$.

(a) Prove: There is a neighborhood of $(0, 0)$ in $x\mu$-space in which all solutions of $f(x, \mu) = 0$ lie on two curves, the first given by $x = 0$, the second given by $x = k(\mu)$, where k is C^1, $k(0) = 0$, and $k'(0) = -\frac{I}{H}$.

(b) Let $g(\mu) = f_x(0, \mu)$. Show that $g(0) = 0$ and $g'(0) = I$. What does this tell us about which of the equilibria $(0, \mu)$ near the origin are attractors and which are repellers?

(c) Let $h(\mu) = f_x(k(\mu), \mu)$ Show that $h(0) = 0$ and $h'(0) = -I$. What does this tell us about which of the equilibria $(k(\mu), \mu)$ near the origin are attractors and which are repellers?

6. Use center manifold reduction to show that

$$\begin{aligned}\dot{x} &= y - 2x,\\ \dot{y} &= \mu + x^2 - y\end{aligned}$$

has a saddle-node bifurcation at $(x, y, \mu) = (1, 2, 1)$. (Suggestion: Shift coordinates to put this point at $(0, 0, 0)$.) Draw the phase portrait on the center manifold near this point, and explain how your picture is related to the full phase portrait near this point.

7. A bead on a rotating hoop satisfies the differential equation

$$\ddot{x} + \dot{x} + \sin x - \mu \sin 2x = 0.$$

Here x is measured in radians from the bottom of the hoop, and the parameter μ is related to the spin rate of the hoop. Letting $y = \dot{x}$, we obtain the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\sin x + \mu \sin 2x - y. \end{aligned}$$

(a) Show that for every μ, $(x, y) = (0, 0)$ is an equilibrium.

(b) Show that the equilibrium at $(x, y) = (0, 0)$ is attracting for $\mu < \frac{1}{2}$, has a 0 eigenvalue for $\mu = \frac{1}{2}$, and is not attracting for $\mu > \frac{1}{2}$.

(c) Use center manifold reduction to show that a pitchfork bifurcation occurs at $\mu = \frac{1}{2}$. Suggestions:

- Let $\lambda = \mu - \frac{1}{2}$.
- Let $y = h(x, \lambda) = x(A + Bx + C\lambda + Dx^2 + \ldots)$. No more terms should be needed.
- Recall that $\sin x = x - \frac{x^3}{3!} + \ldots$.

(d) Are the new equilibria that appear in the pitchfork bifurcation attracting? (Suggestion: begin by looking at the bifurcation diagram.)