MA 732 Homework 4

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1. Let A be an $n \times n$ matrix all of whose eigenvalues have negative real part. Let $g : R^n \to R^n$ be bounded and satisfy a Lipschitz condition, *i.e.*, there is a constant $L > 0$ such that

$$\|g(x_1) - g(x_2)\| \leq L\|x_1 - x_2\|$$

for all x_1 and x_2 in R^n. Let x_0 be a point of R^n. We shall use the Contraction Mapping Theorem to show that if L is small enough, then the solution to the initial value problem

$$\begin{aligned} \dot{x} &= Ax + g(x), \\ x(0) &= x_0, \end{aligned}$$

is defined on the interval $0 \leq t < \infty$ and is bounded.

Our proof begins: If x is a continuous function from $0 \leq t < \infty$ into R^n, define a new continuous function $T(x)$ from $0 \leq t < \infty$ into R^n by the formula

$$T(x)(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}g(x(s))\,ds.$$

   (a) Explain why $T(x)$ is a bounded function of t.

   (b) Show that if L is small enough, then $T : C^0([0,\infty), R^n) \to C^0([0,\infty), R^n)$ is a contraction. How small must L be?

   (c) Explain why the the fixed point of the contraction is a bounded solution of the initial value problem.

2. Show that the differential equation $\dot{x} = -x^5 + c(t)$, where $c(t)$ is a 2π-periodic continuous function, has a 2π-periodic solution. Show that any such solution is asymptotically stable. Use the graph of the Poincaré map to explain why this implies that there is only one 2π-periodic solution.

3. Suppose that $a(t)$ is 2π-periodic with $0 < a(t) < 1$ for all t. Show that the differential equation $\dot{x} = x(x - a(t))(1 - x)$ has at least three 2π-periodic solutions. Hint: Show that $x(t) \equiv 0$ and $x(t) \equiv 1$ are asymptotically stable 2π-periodic solutions, and use the graph of the Poincaré map to explain why this implies that there is a 2π-periodic solution between them.

4. *Fredholm alternative.* Suppose that $a(t)$ and $b(t)$ are 2π-periodic continuous functions, and let $a_0 = \int_0^{2\pi} a(s)\ ds$. Show the following properties of the differential equation $\dot{x} = a(t)x + b(t)$.

   (a) If $a_0 \neq 0$, then there is a unique 2π-periodic orbit. It is asymptotically stable if $a_0 < 0$, and asymptotically unstable if $a_0 > 0$.

   (b) Suppose $a_0 = 0$. Let $c_0 = \int_0^{2\pi} \exp\{\int_s^{2\pi} a(u)\ du\}\, b(s)\, ds$.

      i. If $c_0 = 0$, then every solution is 2π-periodic.

      ii. If $c_0 \neq 0$, then every solution is unbounded.

   Hint: Show using the variation of constants formula that the Poincaré map is

$$P(\xi) = e^{a_0}\xi + \int_0^{2\pi} \exp\{\int_s^{2\pi} a(u)\, du\}\, b(s)\, ds,$$

   and $P(\xi) = \xi$ if and only if $(1 - e^{a_0})\xi = c_0$.

5. *Riccati equation.* Suppose that $a(t)$ and $b(t)$ are 2π-periodic continuous functions. Prove that the Riccati equation

$$\dot{x} = b(t) + a(t)x - x^2$$

   has at most two 2π-periodic solutions. Hint: Suppose that $\phi(t)$ is a 2π-periodic solution. If $x(t)$ is another solution, let $y(t) = x(t) - \phi(t)$. Show that

$$\dot{y} = c(t)y - y^2,$$

   where $c(t) = a(t) - 2\phi(t)$. Then let $w(t) = \frac{1}{y(t)}$. Show that

$$\dot{w} = -c(t)w + 1$$

   Use the Fredholm Alternative to discuss separately the cases $\int_0^{2\pi} c(t)\, dt \neq 0$ and $\int_0^{2\pi} c(t)\ dt = 0$.

6. Recall the Lotka-Volterra system for a prey x and a predator y (formula 1.39 in our text, phase portrait in figure 1.10):

$$\dot{x} = x - xy, \tag{1}$$

$$\dot{y} = \rho(xy - y). \tag{2}$$

   The system (1)–(2) has an infinite number of closed orbits in the first quadrant. However, as the authors point out, this system allows a population to become arbitrarily small yet still recover, which is not realistic. One way to fix this for the prey is to change the growth rate of the prey in the absence of predators from 1 to $\frac{x-\epsilon}{x+\epsilon}$ with $0 < \epsilon < 1$. This growth rate is negative for $x < \epsilon$ and positive for $x > \epsilon$, approaching 1 as x increases. The new system is

$$\dot{x} = x\frac{x - \epsilon}{x + \epsilon} - xy, \tag{3}$$

$$\dot{y} = \rho(xy - y). \tag{4}$$

   In this problem we investigate the asymptotic behavior of solutions of (3)–(4) that begin in the first quadrant. Warning: you may need some help with parts (e) and (h).

(a) Draw the nullclines in the quadrant $x \geq 0$, $y \geq 0$. Indicate the direction of the vector field on the nullclines and in the open regions bounded by the nullclines.

(b) From the vector field you drew, you might think that there are closed orbits, but there are not. Let $F(x, y) =$ right-hand side of (3)–(4), let $g(x, y) = \frac{1}{xy}$. Calculate the divergence of $g(x, y)F(x, y)$, then use Dulac's Criterion to show that there are no closed orbits in the first quadrant.

(c) Find the equilibria with $x \geq 0$ and $y \geq 0$. Use linearization to determine their types (attractor, repeller, or saddle). Don't forget that $0 < \epsilon < 1$.

(d) The complement of the nullclines in the first quadrant consists of four open sets:

$$\begin{aligned} R_1 &= \text{the region where } \dot{x} > 0 \text{ and } \dot{y} > 0\}, \\ R_2 &= \text{the region where } \dot{x} < 0 \text{ and } \dot{y} > 0\}, \\ R_3 &= \text{the region where } \dot{x} < 0 \text{ and } \dot{y} < 0\}, \\ R_4 &= \text{the region where } \dot{x} > 0 \text{ and } \dot{y} < 0\}. \end{aligned}$$

Explain: the portion of the *local* stable manifold of $(\epsilon, 0)$ that lies in the first quadrant is in R_3.

(e) From your sketch of the vector field in part (a), it appears possible that in backward time, the stable manifold of $(\epsilon, 0)$ never leaves R_3; its y-coordinate could increase toward ∞. It also appears that if the stable manifold of $(\epsilon, 0)$ does leave R_3, it might never leaves R_2; its x-coordinate could increase toward ∞. This part and the next show that neither of these things in fact happens.

By part (d) we can choose a point (x_1, y_1) in R_3 that is in the local stable manifold of $(\epsilon, 0)$, with $y_1 << 1$. Consider the compact region K bounded by

(I) the vertical line segment from (x_1, y_1) to (x_1, y_2), with $y_1 << y_2$;

(II) the line segment $y = \frac{\rho}{x_1}(x - x_1) + y_2$ from (x_1, y_2) to $(1, y_3)$ (you could easily calculate y_3);

(III) the portion of the half-parabola $y = y_3 - k(x - 1)^2$, $x \geq 1$, $0 < k << 1$, from $(1, y_3)$ to (x_4, y_4), where (x_4, y_4) is the intersection of the half-parabola with the nullcline $y = \frac{x-\epsilon}{x+\epsilon}$;

(IV) the portion of the nullcline $y = \frac{x-\epsilon}{x+\epsilon}$ from (x_4, y_4) to (x_5, y_1) (note that (x_5, y_1) is the intersection of this nullcline with with the line $y = y_1$); and

(V) the horizontal line segment from (x_5, y_1) to (x_1, y_1).

Show: If y_2 is sufficiently large and k is sufficiently small, then on the boundary of K, the vector field defined by the right-hand side of (3)–(4) points out of K, except along the portion of (IV) with $1 \leq x \leq x_4$.

Suggestion: On line segments (I) and (V), and on the portion of curve (IV) with $x_5 \leq x < 1$, it is obvious that the vector field points out of K. It is equally obvious that on the portion of curve (IV) with $1 \leq x \leq x_4$, the vector field does *not* point out of K. You need to show that if y_2 is sufficiently large and k is sufficiently small, then the vector field points out of K along the curves (II) and (III). For (II), define the function $V(x, y) = y - \frac{\rho}{x_1}(x - x_1)$ and show that for y_2 sufficiently large, $\dot{V} \geq 0$ on the portion of the line $V = y_2$ with $x_1 \leq x \leq 1$. Do something similar for (III). For (III), I suggest that you not calculate x_4. It is enough to note that the half-parabola meets the x-axis at $x = 1 + \sqrt{\frac{y_3}{k}}$, so $1 < x_4 < 1 + \sqrt{\frac{y_3}{k}}$.

(f) Explain: In backwards time, the portion of the stable manifold of $(\epsilon, 0)$ that is in the first quadrant either (i) approaches the equilibrium $(1, \frac{1-\epsilon}{1+\epsilon})$ without leaving R_3, or (ii) passes from R_3 into R_2, then from R_2 into R_1 through the portion of the nullcline $y = \frac{x-\epsilon}{x+\epsilon}$ with $1 < x < x_4$.

(g) Consider the function $L(x, y) = \rho(x - \ln x) + y - \ln y$, which is constant on solutions of (1)–(2) (text, formula 1.40). Show that for the system (3)–(4), $\dot{L} \leq 0$ in the region $x \geq 1$, $y > 0$.

(h) Show that if (x_0, y_0) is in the open first quadrant, then $\{\phi(t, (x_0, y_0)) : t \geq 0\}$ is bounded. There are two cases; do only the case described by (f)(ii). You may assume that the level curves of L are closed and surround $(1, 1)$ as in figure 1.10 in the text.

Suggestion: Let Γ denote the portion of the stable manifold of $(\epsilon, 0)$ from $(\epsilon, 0)$ to its first intersection with the nullcline $y = \frac{x-\epsilon}{x+\epsilon}$, which we denote (x^*, y^*). Let R denote the closed region bounded by Γ, the vertical line segment from (x^*, y^*) to $(x^*, 0)$, and the horizontal line segment from $(x^*, 0)$ to the equilibrium $(\epsilon, 0)$. If $\{\phi(t, (x_0, y_0)) : t \geq 0\}$ lies entirely in R, then it is bounded. If not, there is a time $t_1 \geq 0$ such that the point $(x^\dagger, y^\dagger) = \phi(t_1, (x_0, y_0))$ is not in R. You need to show that $\{\phi(t, (x^\dagger, y^\dagger)) : t \geq 0\}$ is bounded.

(i) Nothing to do on this part, just read it: From part (h), if (x_0, y_0) is in the open first quadrant, then $\omega(x_0, y_0)$ is compact. By a slight generalization of Theorem 6.2.3 in the text, $\omega(x_0, y_0)$ is either a single point, a closed orbit, or a "homoclinic cycle" (defined on p. 182). By part (b) there are no closed orbits, and it is easy to see that there are no homoclinic cycles. Now the interior equilibrium is a repeller, and any solution that approaches $(\epsilon, 0)$ is in its stable manifold. Conclusion: if (x_0, y_0) (i) lies in the open first quadrant, (ii) is not the interior equilibrium, and (iii) is not in the stable manifold of the saddle $(\epsilon, 0)$, then $\phi(t, (x_0, y_0)) \to (0, 0)$ as $t \to \infty$.