MA 732 Homework 3

S. Schecter

February 1, 2013

1. Consider the inhomogeneous linear system

$$\begin{aligned}\dot{x} &= -2x + 4,\\ \dot{y} &= y + \sin t.\end{aligned}$$

   Use the linear lemma that we used in the proof of the stable manifold theorem to find the unique solution $(x(t), y(t))$ that satisfies (i) $x(0) = x_0$ and (2) $(x(t), y(t))$ is bounded on $0 \le t < \infty$. What is $y(0)$?

2. To prove the Stable Manifold Theorem, which was stated carefully in lecture, we considered a differential equation $\dot{x} = Ax + g(x)$, where A and g had certain properties; defined a linear operator T and a nonlinear operator N; proved certain properties of these operators; defined a a C^1 map $F : E^s \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to C^0(\mathbb{R}_+, \mathbb{R}^n)$ by $F(x_s, x)(t) = e^{tA}x_s + (T \circ N)(x)(t)$; showed (roughly speaking) that $F(x_s, \cdot)$ is a contraction; and used the Contraction Mapping Theorem with Parameters to study the fixed point $x = g(x_s)$. Here is a different approach that uses a different map F and uses the Implicit Function Theorem (see Homework 2) instead of the Contraction Mapping Theorem with Parameters. Define $F : E^s \times C^0(\mathbb{R}_+, \mathbb{R}^n) \to C^0(\mathbb{R}_+, \mathbb{R}^n)$ by $F(x_s, x)(t) = x(t) - \big(e^{tA}x_s + (T \circ N)(x)(t)\big)$. Notice that F is C^1.

   (a) Show that $F(0, 0) = 0$.

   (b) Show that $D_2F(0, 0)$ is invertible.

   (c) By the Implicit Function Theorem there is a C^1 function $x = \psi(x_s)$ with $\psi(0) = 0$ such that $F(x_s, \psi(x_s)) = 0$. Use the detailed statement of the Implicit Function Theorem in Homework 4 and study of the map ψ to complete the proof. Feel free to use properties of T and N that were proved in lecture.

3. Use center manifold reduction to draw the phase portrait near the origin for the system

$$\begin{aligned}\dot{x} &= y^2 - 2x^4,\\ \dot{y} &= y - x^2.\end{aligned}$$

4. Use center manifold reduction to draw the phase portrait near the origin for the system

$$\begin{aligned}\dot{x} &= y - x^2,\\ \dot{y} &= -y + x^2 + xy.\end{aligned}$$

5. Use center manifold reduction to describe the phase portrait near the origin for the system

$$\begin{aligned}\dot{x} &= x - z^2 + y^2,\\ \dot{y} &= -y + x^2 + z^2,\\ \dot{z} &= x^2 + y^2.\end{aligned}$$

Suggestion: The eigenvalues are 1, -1, and 0. The center subspace is the z-axis. The center manifold is therefore

$$x = g(z) = Az^2 + \dots, \quad y = h(z) = Bz^2 + \dots.$$

Find A and B, and use them to find the differential equation on the center manifold to fourth order. This should enable you to describe the phase portrait near the origin, which is determined by the flow on the center manifold and the fact that there are both a stable manifold and an unstable manifold.