MA 732 Homework 1

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1. Consider the system

$$\dot{x} = xy, \quad (1)$$
$$\dot{y} = x^2 + 2y^2 + x^3. \quad (2)$$

   If we think of this as $\dot{X} = F(X)$, then $F(0) = 0$ and $DF(0) = 0$. Thus linearization does not help us to figure out the nature of the equilibrium at the origin.

   (a) Convert to polar coordinates using $x = r\cos\theta$, $y = r\sin\theta$. Recall from last semester that when you do this,

$$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y}), \quad \dot{\theta} = \frac{1}{r^2}(x\dot{y} - y\dot{x}).$$

   (b) The system you found in part (a) should have $\dot{r} = \dot{\theta} = 0$ when $r = 0$ because of a factor of r^2 in the $\dot{r}$ equation and a factor of r in the $\dot{\theta}$ equation. Divide your system by r. In the region $r > 0$ this only changes the length of vectors, not their direction.

   (c) The system you found in part (b) still should have a factor of r in the $\dot{r}$ equation, so the circle $r = 0$ is invariant. Use the $\dot{\theta}$ equation you found in part (b) to draw the phase portrait of this system on the circle $r = 0$, locating the equilibria on that circle in the process.

   (d) For the full system of part (b), determine the types of the equilibria on the circle $r = 0$ by linearization.

   (e) Use the information from parts (c) and (d) to draw the phase portrait near the circle $r = 0$ of the system you found in part (b).

   (f) Use the information from part (e) to draw the phase portrait of the original system (1)–(2) near $(x, y) = (0, 0)$.

2. Consider the system

$$\dot{x} = x^2(y - x), \quad (3)$$
$$\dot{y} = y^2(y - 2x). \quad (4)$$

   As in problem 1 there is an equilibrium at the origin, but linearization does not help to analyze it.

(a) Convert to polar coordinates using $x = r\cos\theta$, $y = r\sin\theta$.

(b) The system you found in part (a) should have $\dot{r} = \dot{\theta} = 0$ when $r = 0$ because of a factor of r^3 in the $\dot{r}$ equation and a factor of r^2 in the $\dot{\theta}$ equation. Divide your system by r^2.

(c) The system you found in part (b) still should have a factor of r in the $\dot{r}$ equation, so the circle $r = 0$ is invariant. Use the $\dot{\theta}$ equation you found in part (b) to draw the phase portrait of this system on the circle $r = 0$, locating the equilibria on that circle in the process. Suggestion: Show that you can rewrite the $\dot{\theta}$ equation as

$$\dot{\theta} = \frac{1}{2}\cos\theta\sin\theta(2 - 3\sin 2\theta).$$

The equation $2 - 3\sin 2\theta = 0$ has four solutions in the interval $0 \le \theta < 2\pi$. Thus in all there are eight equilibria on the circle. The direction of flow changes sign at each equilibrium (you don't need to check this).

(d) For the full system of part (b), determine the types of the equilibria on the circle $r = 0$ by linearization. Suggestion: Look at the picture in part (c). Each equilibrium that is attracting on the circle has a negative eigenvalue, and each equilibrium that is repelling on the circle has a positive eigenvalue (you don't need to check this). The other eigenvalue is given by

$$\frac{\partial \dot{r}}{\partial r} = \cos^3\theta\sin\theta - \cos^4\theta + \sin^4\theta - 2\cos\theta\sin^3\theta.$$

You may want to just evaluate this numerically at the four equilibria where it is not easy to evaluate.

(e) Use the information from parts (c) and (d) to draw the phase portrait near the circle $r = 0$ of the system you found in part (b).

(f) Use the information from part (e) to draw the phase portrait of the original system (3)–(4) near $(x, y) = (0, 0)$.

3. Consider the system

$$\dot{x} = xy - x^2y + y^3, \tag{5}$$
$$\dot{y} = y^2 + x^3 - xy^2. \tag{6}$$

As in problem 1 there is an equilibrium at the origin, but linearization does not help to analyze it.

(a) Convert to polar coordinates using $x = r\cos\theta$, $y = r\sin\theta$.

(b) The system you found in part (a) should have $\dot{r} = \dot{\theta} = 0$ when $r = 0$ because of a factor of r^2 in *both* the $\dot{r}$ equation and the $\dot{\theta}$ equations. Divide your system by r^2. Now $\dot{r}$ and $\dot{\theta}$ depend only on θ! The circle $r = 0$ is *not* invariant: many solutions cross it.

(c) For the system of part (b), find the regions where $\dot{r} > 0$, $\dot{r} < 0$, $\dot{\theta} > 0$, $\dot{\theta} < 0$. They are separated by "nullclines," i.e., curves (in this case lines) where $\dot{r} = 0$ or $\dot{\theta} = 0$.

(d) Use the information from part (c) to draw the phase portrait of the system you found in part (b).

(e) Use the information from part (d) to draw the phase portrait of the original system (5)–(6). Remember: your system in part (b) had solutions that crossed $r = 0$, but the origin is an equilibrium for the original system.

4. Consider the differential equation

$$\begin{aligned} \dot{x} &= x^2 + y^2 - 1, && (7) \\ \dot{y} &= 5(xy - 1). && (8) \end{aligned}$$

(a) Show that there are no equilibria.

(b) Use the method of nullclines to draw the phase portrait in the finite plane. (Draw the curves where $\dot{x} = 0$ and $\dot{y} = 0$. Determine the signs of $\dot{x}$ and $\dot{y}$ in the regions between these curves. Use this information to draw the phase portrait.)

(c) Show that there is a unique orbit Γ_+ for which $\frac{y}{x} \to 0$ as $x \to \infty$, and a unique orbit Γ_- for which $\frac{y}{x} \to 0$ as $x \to -\infty$. Suggestion: use the coordinates $u = \frac{1}{x}$, $v = \frac{y}{x}$.

(d) On Γ_+, what does $\frac{y}{x}$ approach as $x \to -\infty$? On Γ_-, what does $\frac{y}{x}$ approach as $x \to \infty$? Try to answer these questions by combining information from parts (b) and (c).

5. Define $F : C^0([a,b], R) \to R$ by $F(\phi) = \int_a^b (\phi(t))^2 \, dx$. *Using the definition of derivative*, prove that F is C^1, and $DF(\phi)\psi = \int_a^b 2\phi(t)\psi(t)\, dt$.

6. Let $Y_1, \ldots, Y_k$ by Banach spaces. Here you'll fill in some details of things we said in class.

(a) Show that $Y_1 \times \ldots \times Y_k$, with the norm $\|(y_1, \ldots, y_k)\| = \max_{1 \le i \le k} \|y_i\|_i$, is a Banach space. (In other words, show that the "norm" just defined is really a norm, and that $Y_1 \times \ldots \times Y_k$ with this norm is complete.)

(b) Define $\Pi_i : Y_1 \times \ldots \times Y_k \to Y_i$ by $\Pi_i(y_1, \ldots, y_k) = y_i$. Show that Π_i is a bounded linear map with norm 1.

(c) Let X be another Banach space, and let $f : X \to Y_1 \times \ldots \times Y_k$ be a map, $f(x) = (f_1(x), \ldots, f_k(x))$. Show that f is continuous if and only if each f_i is continuous. (It may simplify things to note that $f_i = \Pi_i \circ f$.)

(d) Again let X be another Banach space, and let $f : X \to Y_1 \times \ldots \times Y_k$ be a map, $f(x) = (f_1(x), \ldots, f_k(x))$. Assume that each f_i is differentiable at a point $x \in X$. Show that f is differentiable at x, and

$$Df(x)h = (Df_1(x)h, \ldots, Df_k(x)h).$$