Exercise 8.16. Suppose that $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with exactly one zero eigenvalue. Show that there is a nonzero "left eigenvector" $w \in \mathbb{R}^n$ such that $w^{\mathrm{T}}A = 0$. Also, show that v is in the range of A if and only if $\langle v, w\rangle = 0$. Discuss how this exercise gives a method to verify the hypotheses of Theorem 8.12.

Exercise 8.17. Verify the existence of a saddle-node bifurcation for the function $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ given by

$$f(x, y, \lambda) = (\lambda - x^2, -y).$$

Exercise 8.18. Determine the bifurcation diagram for the phase portrait of the differential equation

$$x\ddot{x} + a\dot{x}^2 = b$$

where a and b are parameters.

Exercise 8.19. [Hamiltonian saddle-node] Suppose that

$$\dot{u} = f(u, \lambda), \qquad u \in R^2 \tag{8.12}$$

is a planar *Hamiltonian* family with parameter $\lambda \in \mathbb{R}$. Prove that if $f(u_0, \lambda_0) = 0$ and the corresponding linearization at u_0 has a zero eigenvalue, then this eigenvalue has algebraic multiplicity two. In particular, a planar Hamiltonian system cannot have a saddle-node. Define (u_0, λ_0) to be a *Hamiltonian saddle-node* at λ_0 if $f(u_0, \lambda_0) = 0$ and $f_u(u_0, \lambda_0)$ has a zero eigenvalue with geometric multiplicity one. A *Hamiltonian saddle-node bifurcation* occurs if the following conditions hold:

- There exist $s_0 > 0$ and a smooth curve γ in $R^2 \times R$ such that $\gamma(0) = (u_0, \lambda_0)$ and $f(\gamma(s)) \equiv 0$ for $|s| < s_0$.
- The curve of critical points γ is quadratically tangent to $R^2 \times \{\lambda_0\}$ at (u_0, λ_0).
- The Lyapunov stability type of the rest points on the curve γ changes at $s = 0$.

Prove the following proposition formulated by Jason Bender [24]: *Suppose that the origin in $\mathbb{R}^2 \times \mathbb{R}$ is a Hamiltonian saddle-node for (8.12) and $k \in R^2$ is a nonzero vector that spans the one-dimensional kernel of the linear transformation $f_u(0,0)$. If the two vectors $f_\lambda(0,0) \in R^2$ and $f_{uu}(0,0)(k,k) \in R^2$ are nonzero and not in the range of $f_u(0,0)$, then a Hamiltonian saddle-node bifurcation occurs at the origin.*

Reformulate the hypotheses of the proposition in terms of the Hamiltonian for the family so that there is no mention of the vector k. Also, discuss the Hamiltonian saddle-node bifurcation for the following model of a pendulum with feedback control

$$\dot{x} = y, \qquad \dot{y} = -\sin x - \alpha x + \beta$$

(see [236]). Generalize the proposition to Hamiltonian systems on $\mathbb{R}^{2n}$. (See [160] for the corresponding result for Poincaré maps at periodic orbits of Hamiltonian systems.)

8.3 Poincaré–Andronov–Hopf Bifurcation

Consider the family of differential equations

$$\dot{u} = F(u, \lambda), \qquad u \in \mathbb{R}^N, \quad \lambda \in \mathbb{R}^M \tag{8.13}$$

where λ is a vector of parameters.

Definition 8.20. An ordered pair $(u_0, \lambda_0) \in \mathbb{R}^N \times \mathbb{R}^M$ consisting of a parameter value λ_0 and a rest point u_0 for the corresponding member of the family (8.13) is called a *Hopf point* if there is a curve C in $\mathbb{R}^N \times \mathbb{R}^M$, called an *associated curve*, that is given by $\epsilon \mapsto (C_1(\epsilon), C_2(\epsilon))$ and satisfies the following properties:

(i) $C(0) = (u_0, \lambda_0)$ and $F(C_1(\epsilon), C_2(\epsilon)) \equiv 0$.

(ii) The linear transformation given by the derivative $F_u(C_1(\epsilon), C_2(\epsilon)) : \mathbb{R}^N \to \mathbb{R}^N$ has a pair of nonzero complex conjugate eigenvalues $\alpha(\epsilon) \pm \beta(\epsilon)\, i$, each with algebraic (and geometric) multiplicity one. Also, $\alpha(0) = 0$, $\alpha'(0) \neq 0$, and $\beta(0) \neq 0$.

(iii) Except for the eigenvalues $\pm\beta(0)\, i$, all other eigenvalues of $F_u(u_0, \lambda_0)$ have nonzero real parts.

Our definition says that a one-parameter family of differential equations has a Hopf point if a single pair of complex conjugate eigenvalues, associated with the linearizations of a corresponding family of rest points, crosses the imaginary axis in the complex plane with nonzero speed at the parameter value of the bifurcation point, whereas all other eigenvalues have nonzero real parts. We will show that if some additional generic assumptions are met, then there are members of the family (8.13) that have a limit cycle near the Hopf point.

Let us show first that it suffices to consider the bifurcation for a planar family of differential equations associated with the family (8.13).

Because the linear transformation given by the derivative $F_u(u_0, \lambda_0)$ at the Hopf point (u_0, λ_0) has exactly two eigenvalues on the imaginary axis, the results in Chapter 4, especially equation (4.24), can be used to show that there is a center manifold reduction for the family (8.13) that produces a family of planar differential equations

$$\dot{u} = f(u, \lambda), \qquad u \in \mathbb{R}^2, \quad \lambda \in \mathbb{R}^M, \tag{8.14}$$

with a corresponding Hopf point. Moreover, there is a product neighborhood $U \times V \subset \mathbb{R}^N \times \mathbb{R}^M$ of the Hopf point (u_0, λ_0) such that if $\lambda \in V$ and the corresponding member of the family (8.13) has a bounded orbit in U, then this same orbit is an invariant set for the corresponding member of the planar family (8.14). Thus, it suffices to consider the bifurcation of limit cycles from the Hopf point of this associated planar family.

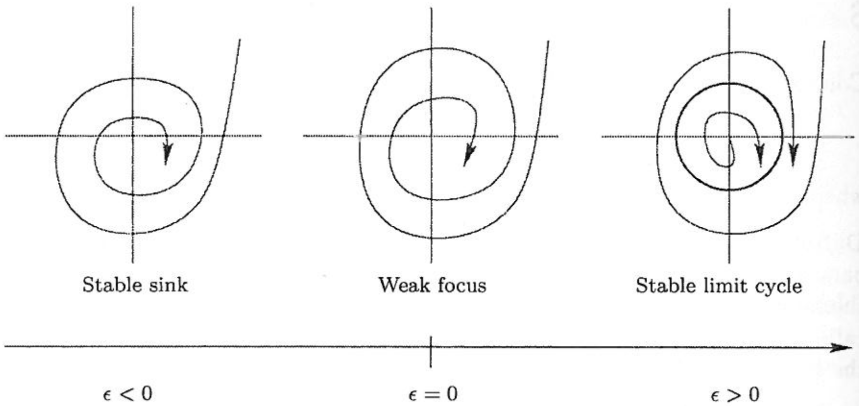

Figure 8.3: Supercritical Hopf bifurcation: A limit cycle emerges from a weak focus as the bifurcation parameter is increased.

There are important technical considerations related to the smoothness and uniqueness of the planar family obtained by a center manifold reduction at a Hopf point. For example, let us note that by the results in Chapter 4 if the family (8.13) is C^1, then the augmented family, obtained by adding a new equation corresponding to the parameters, has a local C^1 center manifold. But this result is not strong enough for the proof of the Hopf bifurcation theorem given below. In fact, we will require the reduced planar system (8.14) to be C^4. Fortunately, the required smoothness can be proved. In fact, using the fiber contraction principle as in Chapter 4, together with an induction argument, it is possible to prove that if $0 < r < \infty$ and the family (8.13) is C^r, then the reduced planar system at the Hopf point is also C^r in a neighborhood of the Hopf point. Let us also note that whereas local center manifolds are not necessarily unique, it turns out that all rest points, periodic orbits, homoclinic orbits, et cetera, that are sufficiently close to the original rest point, are on every center manifold. Thus, the bifurcation phenomena that are determined by reduction to a center manifold do not depend on the choice of the local center manifold (see, for example, [58]).

Let us say that a set $S \subset \mathbb{R}^N$ has *radii* (r_1, r_2) *relative to a point* p if $r_1 \geq 0$ is the radius of the smallest $\mathbb{R}^N$-ball centered at p that contains S and the distance from S to p is $r_2 \geq 0$.

Definition 8.21. The planar family (8.14) has a *supercritical Hopf bifurcation* at a Hopf point with associated curve $\epsilon \mapsto (c_1(\epsilon), c_2(\epsilon))$ if there are three positive numbers ϵ_0, K_1, and K_2 such that for each ϵ in the open interval $(0, \epsilon_0)$ the differential equation $\dot{u} = f(u, c_2(\epsilon))$ has a hyperbolic limit cycle with radii

$$(K_1\sqrt{\epsilon} + O(\epsilon),\ K_2\sqrt{\epsilon} + O(\epsilon))$$

relative to the rest point $u = c_1(\epsilon)$. The bifurcation is called *subcritical* if there is a similar limit cycle for the systems with parameter values in the range $-\epsilon_0 < \epsilon < 0$. Also, we say that the family (8.13) has a supercritical (respectively, subcritical) Hopf bifurcation at a Hopf point if the corresponding (center manifold) reduced system (8.14) has a supercritical (respectively, subcritical) Hopf bifurcation.

To avoid mentioning several similar cases as we proceed, let us consider only Hopf points such that the parametrized eigenvalues $\alpha \pm \beta i$ satisfy the additional assumptions

$$\alpha'(0) > 0, \qquad \beta(0) > 0. \tag{8.15}$$

In particular, we will restrict attention to the supercritical Hopf bifurcation as depicted in Figure 8.3.

Under our standing hypothesis (8.15), a rest point on the associated curve $\epsilon \mapsto c(\epsilon)$ of the Hopf point is a stable hyperbolic focus for the corresponding system (8.14) for $\epsilon < 0$ and an unstable hyperbolic focus for $\epsilon > 0$. We will introduce an additional hypothesis that implies "weak attraction" toward the rest point u_0 at the parameter value λ_0. In this case, there is a stable limit cycle that "bifurcates from this rest point" as ϵ increases through $\epsilon = 0$. This change in the qualitative behavior of the system as the parameter changes is the bifurcation that we wish to describe, namely, the supercritical Hopf bifurcation.

Before defining the notion of weak attraction, we will simplify the family (8.14) by a local change of coordinates and a reduction to one-parameter. Note that, after the translation $v = u - c_1(\epsilon)$, the differential equation (8.14) becomes

$$\dot{v} = f(v + c_1(\epsilon), \lambda)$$

with $f(0 + c_1(\epsilon), c_2(\epsilon)) \equiv 0$. In particular, in the new coordinates, the associated rest points remain at the origin for all values of the parameter ϵ. Thus, it suffices to consider the family (8.14) to be of the form

$$\dot{u} = f(u, \lambda), \qquad u \in \mathbb{R}^2, \quad \lambda \in \mathbb{R}, \tag{8.16}$$

only now with a Hopf point at $(u, \lambda) = (0, 0) \in \mathbb{R}^2 \times \mathbb{R}$ and with the associated curve c given by $\lambda \mapsto (0, \lambda)$.

Proposition 8.22. *If $(u, \lambda) = (0, 0) \in \mathbb{R}^2 \times \mathbb{R}$ is a Hopf point for the family (8.16) with associated curve $\lambda \mapsto (0, \lambda)$ and eigenvalues $\alpha(\lambda) \pm \beta(\lambda) i$, then there is a smooth parameter-dependent linear change of coordinates of the form $u = L(\lambda)z$ that transforms the system matrix $A(\lambda) := f_u(0, \lambda)$ of the linearization at the origin along the associated curve into the Jordan normal form*

$$\begin{pmatrix} \alpha(\lambda) & -\beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix}.$$

Proof. Suppose that $w(\lambda) = u_1(\lambda) + u_2(\lambda)\, i$ is a (nonzero) eigenvector for the eigenvalue $\alpha(\lambda) + \beta(\lambda)\, i$. We will show that there is an eigenvector of the form

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} v_1(\lambda) \\ v_2(\lambda) \end{pmatrix} i.$$

To prove this fact, it suffices to find a family of complex numbers $c(\lambda) + d(\lambda)\, i$ such that

$$(c + d\,i)(u_1 + u_2\, i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} i$$

for a family of numbers $v_1, v_2 \in \mathbb{R}$ where the minus sign is inserted to determine a convenient orientation. Equivalently, it suffices to solve the equation

$$cu_1 - du_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which is expressed in matrix form as follows:

$$(u_1, -u_2)\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since the eigenvectors w and $\bar{w}$ corresponding to the distinct eigenvalues $\alpha \pm \beta\, i$ are linearly independent and

$$(u_1, -u_2)\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = (w, \bar{w}),$$

it follows that $\det\,[u_1, -u_2] \neq 0$, and therefore we can solve (uniquely) for the vector (c, d).

Using this fact, we have the eigenvalue equation

$$A\Big(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big) = (\alpha + i\beta)\Big(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big),$$

as well as its real and imaginary parts

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{8.17}$$

Hence, if

$$L := \begin{pmatrix} 1 & v_1 \\ 0 & v_2 \end{pmatrix},$$

then

$$AL = L\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Again, since the vectors u_1 and u_2 are linearly independent, so are the following nonzero scalar multiples of these vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Thus, we have proved that the matrix L is invertible. Moreover, we can solve explicitly for v_1 and v_2. Indeed, using the equations (8.17), we have

$$(A - \alpha I)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

If we now set

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then

$$v_1 = \frac{a_{11} - \alpha}{\beta}, \qquad v_2 = \frac{a_{21}}{\beta}.$$

Here $\beta := \beta(\lambda)$ is not zero at $\lambda = 0$, so the functions $\lambda \mapsto v_1(\lambda)$ and $\lambda \mapsto v_2(\lambda)$ are smooth. Finally, the change of coordinates $v = L(\lambda)z$ transforms the family of differential equations (8.16) to $\dot{z} = L^{-1}(\lambda)f(L(\lambda)z, \lambda)$, and the linearization of the transformed equation at $z = 0$ is given by

$$\begin{pmatrix} \alpha(\lambda) & -\beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix}.$$

The matrix function $\lambda \mapsto L^{-1}(\lambda)$ is also smooth at the origin. It is given by

$$L^{-1} = \frac{1}{v_2}\begin{pmatrix} v_2 & -v_1 \\ 0 & 1 \end{pmatrix}$$

where $1/v_2(\lambda) = \beta(\lambda)/a_{21}(\lambda)$. But, if $a_{21}(\lambda) = 0$, then the linearization has real eigenvalues, in contradiction to our hypotheses. □

By Proposition 8.22, there is no loss of generality if we assume that the differential equation (8.16) has the form

$$\begin{aligned} \dot{x} &= \alpha(\lambda)x - \beta(\lambda)y + g(x, y, \lambda), \\ \dot{y} &= \beta(\lambda)x + \alpha(\lambda)y + h(x, y, \lambda) \end{aligned} \tag{8.18}$$

where the functions g and h together with their first partial derivatives with respect to the space variables vanish at the origin; the real functions $\lambda \mapsto \alpha(\lambda)$ and $\lambda \mapsto \beta(\lambda)$ are such that $\alpha(0) = 0$ (the real part of the linearization must vanish at $\lambda = 0$) and, by our standing assumption, $\alpha'(0) > 0$ (the derivative of the real part does not vanish at $\lambda = 0$); and, by the assumption that $\beta(0) > 0$, the eigenvalues $\alpha(\lambda) \pm i\beta(\lambda)$ are nonzero complex conjugates for $|\lambda|$ sufficiently close to zero. Moreover, there is no loss of generality if we assume that $\beta(0) = 1$. Indeed, this normalization can be achieved by a reparametrization of time in the family (8.18).

We will seek a periodic orbit of the family (8.18) near the origin of the coordinate system by applying the implicit function theorem to find a zero of the associated displacement function that is defined along the x-axis. For this application of the implicit function theorem we have to check that the displacement function has a smooth extension to the origin. While it is clear that the displacement has a continuous extension to the origin—define its value at the rest point to be zero—it is not clear that the extended displacement function is smooth. Indeed, the proof that the return map exists near a point p on a Poincaré section is based on the implicit function theorem and requires that the vector field be transverse to the section at p. But this condition is not satisfied at the origin for members of the family (8.18) because the vector field vanishes at this rest point.

Let us show that the displacement function for the system (8.18) is indeed smooth by using the blowup construction discussed in Section 1.8.5. The idea is that we can bypass the issue of the smoothness of the displacement at the origin for the family (8.18) by blowing up at the rest point. In fact, by changing the family (8.18) to polar coordinates we obtain the family

$$\dot{r} = \alpha(\lambda)r + p(r,\theta,\lambda), \qquad \dot{\theta} = \beta(\lambda) + q(r,\theta,\lambda) \tag{8.19}$$

where

$$p(r,\theta,\lambda) := g(r\cos\theta, r\sin\theta,\lambda)\cos\theta + h(r\cos\theta, r\sin\theta,\lambda)\sin\theta,$$
$$q(r,\theta,\lambda) := \frac{1}{r}\big(h(r\cos\theta, r\sin\theta,\lambda)\cos\theta - g(r\cos\theta, r\sin\theta,\lambda)\sin\theta\big).$$

Since $(x,y) \mapsto g(x,y,\lambda)$ and $(x,y) \mapsto h(x,y,\lambda)$ and their first partial derivatives vanish at the origin, the function q in system (8.19) has a removable singularity at $r = 0$. Thus, the system is smooth. Moreover, by the change to polar coordinates, the rest point at the origin in the plane has been blown up to the circle $\{0\} \times \mathbb{T}$ on the phase cylinder $\mathbb{R} \times \mathbb{T}$. In our case, where $\beta(\lambda) \neq 0$, the rest point at the origin (for each choice of that parameter λ) corresponds to the periodic orbit on the cylinder given by the solution $r(t) \equiv 0$ and $\theta(t) = \beta(\lambda)t$. A Poincaré section on the cylinder for these periodic orbits, for example the line $\theta = 0$, has a smooth (parametrized) return map that is equivalent to the corresponding return map on the x-axis for the family (8.18). Thus, if we blow down—that is, project back to the plane—then the image of our transversal section is a smooth section for the flow with a smooth return map and a smooth return time map. In particular, both maps are smooth at the origin. In other words, the displacement function on the x-axis of the plane is conjugate (by the change of coordinates) to the smooth displacement function defined on the line $\theta = 0$ in the cylinder.

We will take advantage of the geometry on the phase cylinder: There our bifurcation problem concerns the bifurcation of periodic orbits *from a periodic orbit* rather than the bifurcation of periodic orbits from a rest point. Indeed, Hopf bifurcation on the phase cylinder is analogous to bifurcation from a multiple limit cycle as in our previous discussion following the Weierstrass preparation theorem (Theorem 5.15) on page 384.

For the generic case, we will soon see that the limit cycle, given on the cylinder by the set $\{(r,\theta) : r = 0\}$ for the family (8.19) at $\lambda = 0$, has multiplicity three. But, unlike the general theory for bifurcation from a multiple limit cycle with multiplicity three, the Hopf bifurcation has an essential new feature revealed by the geometry of the blowup: The bifurcation is symmetric with respect to the set $\{(r,\theta) : r = 0\}$. More precisely, each member of the family (8.19) is invariant under the change of coordinates given by

$$R = -r, \qquad \Theta = \theta - \pi. \tag{8.20}$$

While this symmetry has many effects, it should at least be clear that if a member of the family (8.19) has a periodic orbit that does not coincide with the set $\{(r,\theta) : r = 0\}$, then the system has two periodic orbits: one in the upper half cylinder, and one in the lower half cylinder. Also, if the set $\{(r,\theta) : r = 0\}$ is a limit cycle, then it cannot be semistable, that is, attracting on one side and repelling on the other (see Exercise 8.23). The geometry is similar to the geometry of the pitchfork bifurcation (see Exercise 8.11 and Section 8.4).

The general theory of bifurcations with symmetry is an important topic that is covered in detail in the excellent books [97] and [98].

Exercise 8.23. Prove: If the set $\Gamma := \{(r,\theta) : r = 0\}$ on the cylinder is a limit cycle for the member of the family (8.19) at $\lambda = 0$, then this limit cycle is not semistable. State conditions that imply Γ is a limit cycle and conditions that imply it is a hyperbolic limit cycle.

By our hypotheses, if $|r|$ is sufficiently small, then the line $\{(r,\theta) : \theta = 0\}$ is a transversal section for the flow of system (8.19) on the phase cylinder. Moreover, as we have mentioned above, there is a smooth displacement function defined on this section. In fact, let $t \mapsto (r(t,\xi,\lambda), \theta(t,\xi,\lambda))$ denote the solution of the differential equation (8.19) with the initial condition

$$r(0,\xi,\lambda) = \xi, \qquad \theta(0,\xi,\lambda) = 0,$$

and note that (because $\beta(0) = 1$)

$$\theta(2\pi,0,0) = 2\pi, \qquad \dot{\theta}(2\pi,0,0) = \beta(0) \neq 0.$$

By an application of the implicit function theorem, there is a product neighborhood $U_0 \times V_0$ of the origin in $\mathbb{R}\times\mathbb{R}$, and a function $T : U_0 \times V_0 \to \mathbb{R}$ such that $T(0,0) = 2\pi$ and $\theta(T(\xi,\lambda),\xi,\lambda) \equiv 2\pi$. Thus, the desired displacement function $\delta : U_0 \times V_0 \to \mathbb{R}$ is defined by

$$\delta(\xi,\lambda) := r(T(\xi,\lambda),\xi,\lambda) - \xi. \tag{8.21}$$

The displacement function (8.21) is complicated by the presence of the implicitly defined return-time function T, a difficulty that can be avoided by yet another change of coordinates. Indeed, since $T(0,0) = 2\pi$ and $\dot\theta(t,0,0) = \beta(0) \neq 0$, it follows from the continuity of the functions T and $\dot\theta$ and the implicit function theorem that there is a product neighborhood $U \times V$ of the origin with $U \times V \subseteq U_0 \times V_0$ such that for each $(\xi,\lambda) \in U \times V$ the function $t \mapsto \theta(t,\xi,\lambda)$ is invertible on some bounded time interval containing $T(\xi,\lambda)$ (see Exercise 8.28). Moreover, if the inverse function is denoted by $s \mapsto \theta^{-1}(s,\xi,\lambda)$, then the function $\rho : \mathbb{R}\times U\times V \to \mathbb{R}$ defined by

$$\rho(s,\xi,\lambda) = r(\theta^{-1}(s,\xi,\lambda),\xi,\lambda)$$

is a solution of the initial value problem

$$\frac{d\rho}{ds} = \frac{\alpha(\lambda)\rho + p(\rho,s,\lambda)}{\beta(\lambda) + q(\rho,s,\lambda)}, \qquad \rho(0,\xi,\lambda) = \xi$$

and

$$\rho(2\pi,\xi,\lambda) = r(T(\xi,\lambda),\xi,\lambda).$$

If we rename the variables ρ and s to *new variables* r and θ, then the displacement function $\delta : U \times V \to \mathbb{R}$ as defined in equation (8.21) with respect to the original variable r is also given by the formula

$$\delta(\xi,\lambda) = r(2\pi,\xi,\lambda) - \xi \tag{8.22}$$

where $\theta \mapsto r(\theta,\xi,\lambda)$ is the solution of the initial value problem

$$\frac{dr}{d\theta} = \frac{\alpha(\lambda)r + p(r,\theta,\lambda)}{\beta(\lambda) + q(r,\theta,\lambda)}, \qquad r(0,\xi,\lambda) = \xi. \tag{8.23}$$

In particular, with respect to the differential equation (8.23), the "return time" does not depend on the position ξ along the Poincaré section or the value of the parameter λ; rather, it has the constant value 2π.

Definition 8.24. Suppose that $(u,\lambda) = (0,0) \in \mathbb{R}^2 \times \mathbb{R}$ is a Hopf point for the family (8.16). The corresponding rest point $u = 0$ is called a *weak attractor* (respectively, a *weak repeller*) if the associated displacement function (8.22) is such that $\delta_{\xi\xi\xi}(0,0) < 0$ (respectively, $\delta_{\xi\xi\xi}(0,0) > 0$). In addition, the Hopf point $(u,\lambda) = (0,0)$ is said to have *multiplicity one* if $\delta_{\xi\xi\xi}(0,0) \neq 0$.

Theorem 8.25 (Hopf Bifurcation Theorem). *If the family of differential equations (8.16) has a Hopf point at $(u,\lambda) = (0,0) \in \mathbb{R}^2 \times \mathbb{R}$ and the corresponding rest point at the origin is a weak attractor (respectively, a weak repeller), then there is a supercritical (respectively, subcritical) Hopf bifurcation at this Hopf point.*

Proof. Let us assume that the family (8.16) is C^4. By Proposition 8.22, there is a smooth change of coordinates that transforms the family (8.16) into the family (8.18). Moreover, because $\beta(0) \neq 0$, the function

$$S(r,\theta,\lambda) := \frac{\alpha(\lambda)r + p(r,\theta,\lambda)}{\beta(\lambda) + q(r,\theta,\lambda)},$$

and therefore the family of differential equations

$$\frac{dr}{d\theta} = S(r,\theta,\lambda), \tag{8.24}$$

is as smooth as the original differential equation (8.16); that is, it is at least in class C^4.

The associated displacement function δ defined in equation (8.22) is given by the C^4 function

$$\delta(\xi,\lambda) := r(2\pi,\xi,\lambda) - \xi \tag{8.25}$$

where $\theta \mapsto r(\theta,\xi,\lambda)$ is the solution of the differential equation (8.24) with initial condition $r(0,\xi,\lambda) = \xi$. Moreover, each function $\xi \mapsto \delta(\xi,\lambda)$ is defined in a neighborhood of $\xi = 0$ in $\mathbb{R}$.

Since $\delta(0,\lambda) \equiv 0$, the displacement function is represented as a series,

$$\delta(\xi,\lambda) = \delta_1(\lambda)\xi + \delta_2(\lambda)\xi^2 + \delta_3(\lambda)\xi^3 + O(\xi^4),$$

whose first-order coefficient is given by

$$\delta_1(\lambda) = \delta_\xi(0,\lambda) = r_\xi(2\pi,0,\lambda) - 1$$

where $\theta \mapsto r_\xi(\theta,0,\lambda)$ is the solution of the variational initial value problem

$$\frac{dr_\xi}{d\theta} = S_r(0,\theta,\lambda)r_\xi = \frac{\alpha(\lambda)}{\beta(\lambda)}r_\xi, \qquad r_\xi(0,0,\lambda) = 1.$$

Hence, by solving the scalar first order linear differential equation, we have that

$$\delta_1(\lambda) = r_\xi(2\pi,0,\lambda) - 1 = e^{2\pi\alpha(\lambda)/\beta(\lambda)} - 1.$$

Moreover, since $\alpha(0) = 0$, it follows that

$$\delta(\xi,0) = \xi^2\big(\delta_2(0) + \delta_3(0)\xi + O(\xi^2)\big).$$

Note that if $\delta_2(0) \neq 0$, then $\delta(\xi, 0)$ has constant sign for sufficiently small $|\xi| \neq 0$, and therefore the trajectories of the corresponding system (8.19) at $\lambda = 0$ do not spiral around the origin of its phase plane (draw a picture); equivalently, the periodic orbit $\{(r, \theta) : r = 0\}$ on the phase cylinder is a semistable limit cycle. But using the assumptions that $\alpha(0) = 0$ and $\beta(0) \neq 0$ and Exercise 8.23, this qualitative behavior cannot occur. In particular, the existence of a semistable limit cycle on the phase cylinder violates the symmetry (8.20), which carries over to the differential equation (8.23). In fact, if $\theta \to r(\theta, \xi, \lambda)$ is a solution of equation (8.23), then so is the function $\theta \to -r(\theta + \pi, \xi, \lambda)$. For all of these equivalent reasons, we have that $\delta_2(0) = 0$.

Consider the function $\Delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined on the domain of the displacement function by

$$\Delta(\xi, \lambda) = \delta_1(\lambda) + \delta_2(\lambda)\xi + \delta_3(\lambda)\xi^2 + O(\xi^3),$$

and note that

$$\begin{aligned}
\Delta(0, 0) &= e^{2\pi\alpha(0)/\beta(0)} - 1 = 0, \\
\Delta_\xi(0, 0) &= \delta_2(0) = 0, \\
\Delta_{\xi\xi}(0, 0) &= 2\delta_3(0) = \delta_{\xi\xi\xi}(0, 0)/3 \neq 0, \\
\Delta_\lambda(0, 0) &= 2\pi\alpha'(0)/\beta(0) > 0.
\end{aligned}$$

By Proposition 8.2, the function Δ has a saddle-node bifurcation at $\xi = 0$ for the parameter value $\lambda = 0$. In particular, there is a curve $\xi \mapsto (\xi, \gamma(\xi))$ in $\mathbb{R} \times \mathbb{R}$ with $\gamma(0) = 0$, $\gamma'(0) = 0$, and $\gamma''(0) \neq 0$ such that $\Delta(\xi, \gamma(\xi)) \equiv 0$. As a result, we have that

$$\delta(\xi, \gamma(\xi)) = \xi\Delta(\xi, \gamma(\xi)) \equiv 0,$$

and therefore if $\lambda = \gamma(\xi)$, then there is a periodic solution of the corresponding member of the family (8.18) that meets the Poincaré section at the point with coordinate ξ.

For the remainder of the proof, let us assume that $\delta_{\xi\xi\xi}(0, 0) < 0$; the case where $\delta_{\xi\xi\xi}(0, 0) > 0$ is similar.

By Proposition 8.2, we have the inequality

$$\gamma''(0) = -\frac{\Delta_{\xi\xi}(0, 0)}{\Delta_\lambda(0, 0)} = -\frac{\beta(0)}{6\pi\alpha'(0)}\delta_{\xi\xi\xi}(0, 0) > 0,$$

and therefore the coefficient of the leading-order term of the series

$$\lambda = \gamma(\xi) = \frac{\gamma''(0)}{2}\xi^2 + O(\xi^3)$$

does not vanish. Hence, the position coordinate $\xi > 0$ corresponding to a periodic solution is represented as follows by a power series in $\sqrt{\lambda}$:

$$\xi = \left(-\lambda\frac{12\pi\alpha'(0)}{\beta(0)\delta_{\xi\xi\xi}(0, 0)}\right)^{1/2} + O(\lambda). \tag{8.26}$$

Thus, the distance from the periodic orbit to the origin is of the form $K_2\sqrt{k}+O(k)$ for some constant K_2. Using the construction in the discussion following display (8.21), it is easy to see that the function $S = S(r, \theta, \lambda)$ can be restricted to a compact subset of its domain with no loss of generality for our bifurcation analysis. By continuity, the magnitude of the partial derivative S_r is bounded by some constant $K > 0$ on such a compact set. Note that $S(0, \theta, \lambda) \equiv 0$ and apply the mean value theorem to S to obtain the inequality

$$|r(\theta, \xi, \lambda)| = |\xi| + \int_0^\theta |S(r, \phi, \lambda)|\, d\phi \leq |\xi| + K\int_0^\theta |r|\, d\phi.$$

By an application of Gronwall's inequality, we have that

$$|r(\theta, \xi, \lambda)| \leq \xi e^{2\pi K}$$

on our periodic solution. Hence, the periodic solution lies in a ball whose radius is $K_1\sqrt{k} + O(k)$ for some constant K_1, as required.

The proof will be completed by showing that the periodic solution corresponding to ξ given by the equation (8.26) is a stable hyperbolic limit cycle.

Consider the Poincaré map defined by

$$P(\xi, \lambda) := \delta(\xi, \lambda) + \xi = \xi(\Delta(\xi, \lambda) + 1)$$

and note that

$$P_\xi(\xi, \lambda) = \xi\Delta_\xi(\xi, \lambda) + \Delta(\xi, \lambda) + 1.$$

At the periodic solution we have $\lambda = \gamma(\xi)$, and therefore

$$P_\xi(\xi, \gamma(\xi)) = \xi\Delta_\xi(\xi, \gamma(\xi)) + 1.$$

Moreover, because $\Delta(\xi, \gamma(\xi)) \equiv 0$, we have the identity

$$\Delta_\xi(\xi, \gamma(\xi)) = -\Delta_\lambda(\xi, \gamma(\xi))\gamma'(\xi).$$

Using the relations $\Delta_\lambda(0, 0) > 0$, $\gamma'(0) = 0$, and $\gamma''(0) > 0$, it follows that if $\xi > 0$ is sufficiently small, then $\gamma'(\xi) > 0$ and $-\Delta_\lambda(\xi, \gamma(\xi)) < 0$; hence, $\Delta_\xi(\xi, \gamma(\xi)) < 0$ and $0 < P_\xi(\xi, \gamma(\xi)) < 1$. In other words, the periodic solution is a hyperbolic stable limit cycle. □

While the presentation given in this section discusses the most important ideas needed to understand the Hopf bifurcation, there are a few unresolved issues. Note first that sufficient conditions for the Hopf bifurcation are given only for a two-dimensional system obtained by restriction to a center manifold, not for the original system of differential equations. In particular, the definition of a weak attractor is only given for two-dimensional systems. Also, we have not discussed an efficient method to determine the sign of the third space-derivative of the displacement function, an essential step for practical applications of the Hopf bifurcation theorem. For a resolution of the first issue see [151]; the second issue is addressed in the next section.

Exercise 8.26. Consider the two systems

$$\dot{r} = \lambda r \pm r^3, \qquad \dot{\theta} = 1 + ar^2,$$

where (r, θ) are polar coordinates. Show that the + sign system has a supercritical Hopf bifurcation and the − sign system has a subcritical Hopf bifurcation. The given systems are normal forms for the Hopf bifurcation.

Exercise 8.27. Show that the system

$$\dot{x} = \lambda x - y + xy^2, \qquad \dot{y} = x + \lambda y + y^3$$

has a subcritical Hopf bifurcation. Hint: Change to polar coordinates and compute (explicitly) the Poincaré map defined on the positive x-axis. Recall that Bernoulli's equation $\dot{z} = a(t)z + b(t)z^{n+1}$ is transformed to a linear equation by the change of variables $w = z^{-n}$.

Exercise 8.28. Suppose that $K \subseteq \mathbb{R}$ and $W \subseteq \mathbb{R}^k$ are open sets, $g : K \times W \to \mathbb{R}$ is a smooth function, and $T > 0$. If $[0, T] \subset K$, $0 \in W$, and $g_t(t, 0) \neq 0$ for all $t \in K$, then there are open product neighborhoods $I \times U \subseteq K \times W$ and $J \times V \subseteq \mathbb{R} \times \mathbb{R}^k$ with $[0, T] \subset I$ and $0 \in U$ and a smooth function $h : J \times V \to \mathbb{R}$ such that $h(g(t, u), u) = t$ whenever $(t, u) \in I \times U$. Hint: Consider the function $G : K \times W \times \mathbb{R} \to \mathbb{R}$ given by $G(t, w, s) = g(t, w) - s$ and note that $G(t, w, g(t, w)) \equiv 0$ and $G_t(t, w, g(t, w)) \neq 0$. Apply the implicit function theorem to obtain a function h such that $G(h(s, w), w, s) \equiv 0$ and $h(g(t, w), w) = t$. The implicit function is only locally defined but it is unique. Use the uniqueness to show that h is defined globally on an appropriate product neighborhood.

8.3.1 Multiple Hopf Bifurcation

The hypothesis in the Hopf bifurcation theorem, which states that a Hopf point has multiplicity one, raises at least two important questions: How can we check the sign of the third space-derivative $\delta_{\xi\xi\xi}(0, 0)$ of the displacement function? What happens if $\delta_{\xi\xi\xi}(0, 0) = 0$? The answers to these questions will be discussed in this section.

For the second question, let us note that (in the proof of the Hopf bifurcation theorem) the condition $\delta_{\xi\xi\xi}(0, 0) \neq 0$ ensures that the series representation of the displacement function has a nonzero coefficient at the lowest possible order. If this condition is not satisfied because $\delta_{\xi\xi\xi}(0, 0) = 0$, then the Hopf point is called *multiple* and the corresponding Hopf bifurcation is called a *multiple Hopf bifurcation.*

Let us consider the multiple Hopf bifurcation for the case of a planar vector field that depends on a vector of parameters. More precisely, we will

consider the parameter λ in $\mathbb{R}^M$ and a corresponding family of differential equations

$$\dot{u} = f(u, \lambda), \qquad u \in \mathbb{R}^2 \tag{8.27}$$

with the following additional properties: the function f is real analytic; at the parameter value $\lambda = \lambda^*$, the origin $u = 0$ is a rest point for the differential equation $\dot{u} = f(u, \lambda^*)$; and the eigenvalues of the linear transformation $f_u(0, \lambda^*)$ are nonzero pure imaginary numbers. Under these assumptions, the displacement function δ is represented by a convergent power series of the form

$$\delta(\xi, \lambda) = \sum_{j=1}^{\infty} \delta_j(\lambda)\xi^j. \tag{8.28}$$

Definition 8.29. The rest point at $u = 0$, for the member of the family (8.27) at the parameter value $\lambda = \lambda^*$, is called a *weak focus of order k* if k is a positive integer such that

$$\delta_1(\lambda^*) = \cdots = \delta_{2k}(\lambda^*) = 0, \qquad \delta_{2k+1}(\lambda^*) \neq 0.$$

It is not difficult to show—a special case is proved in the course of the proof of the Hopf bifurcation theorem—that if $\delta_1(\lambda^*) = \cdots = \delta_{2k-1}(\lambda^*) = 0$, then $\delta_{2k}(\lambda^*) = 0$. In fact, this is another manifestation of the symmetry given in display (8.20).

The next theorem is a corollary of the Weierstrass preparation theorem (Theorem 5.15).

Proposition 8.30. *If the family (8.27) has a weak focus of order k at $u = 0$ for the parameter value $\lambda = \lambda^*$, then at most k limit cycles appear in a corresponding multiple Hopf bifurcation. More precisely, there is some $\epsilon > 0$ and some $\nu > 0$ such that $\dot{u} = f(u, \lambda)$ has at most k limit cycles in the open set $\{u \in \mathbb{R}^2 : |u| < \nu\}$ whenever $|\lambda - \lambda^*| < \epsilon$.*

While Proposition 8.30 states that at most k limit cycles appear in a multiple Hopf bifurcation at a weak focus of order k, additional information about the set of coefficients $\{\delta_{2j+1}(\lambda) : j = 0, \ldots, k\}$ is required to determine precisely how many limit cycles appear. For example, to obtain the maximum number k of limit cycles, it suffices to have these coefficients be independent in the following sense: There is some $\delta > 0$ such that for each $j \leq k$ and each $\epsilon > 0$, if $|\lambda_0 - \lambda^*| < \delta$ and

$$\delta_1(\lambda_0) = \delta_2(\lambda_0) = \cdots = \delta_{2j-1}(\lambda_0) = 0, \qquad \delta_{2j+1}(\lambda_0) \neq 0,$$

then there is a point λ_1 such that $|\lambda_1 - \lambda_0| < \epsilon$ and

$$\delta_1(\lambda_1) = \cdots = \delta_{2j-3}(\lambda_1) = 0, \qquad \delta_{2j-1}(\lambda_1)\delta_{2j+1}(\lambda_1) < 0.$$