manifold? The unstable manifold? What can you conclude about the dynamics on the center manifold near $x = 0$?

Compare this with the nonHamiltonian case. (*Hint:* to begin, you might want to consider the simplest case, $n = 2$.)

**2.19** Consider the $\mathbf{C}^r$ ($r \geq 1$) map

$$x \mapsto f(x), \qquad x \in \mathbb{R}^n. \tag{E2.14}$$

Suppose that the map has a fixed point at $x = x_0$, i.e.,

$$x_0 = f(x_0).$$

Next consider the vector field

$$\dot{x} = f(x) - x. \tag{E2.15}$$

Clearly (E2.15) has a fixed point, and $x = x_0$. What can you determine about the orbit structure near the fixed point of the map (E2.14) based on knowledge of the orbit structure near the fixed point $x = x_0$ of the vector field (E2.15)?

**2.20** Consider the $\mathbf{C}^r$ map

$$f: \mathbb{R}^1 \to \mathbb{R}^1$$

and denote the Taylor expansion of $f$ by

$$f(x) = a_0 + a_1 x + \cdots + a_{r-1}x^{r-1} + \mathcal{O}(|x|^r).$$

Suppose $f$ is identically zero. Then show that $a_i = 0$, $i = 0, \ldots, r-1$. Does the same result hold for the $\mathbf{C}^r$ map

$$f: \mathbb{R}^n \to \mathbb{R}^n, \qquad n > 1?$$

# 3

# Local Bifurcations

In this chapter we study local bifurcations of vector fields and maps. By the term "local" we mean bifurcations occurring in a neighborhood of a fixed point. The term "bifurcation of a fixed point" will be defined after we have considered several examples. We begin by studying bifurcations of fixed points of vector fields.

## 3.1 Bifurcation of Fixed Points of Vector Fields

Consider the parameterized vector field

$$\dot{y} = g(y, \lambda), \qquad y \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^p, \tag{3.1.1}$$

where $g$ is a $\mathbf{C}^r$ function on some open set in $\mathbb{R}^n \times \mathbb{R}^p$. The degree of differentiability will be determined by our need to Taylor expand (3.1.1). Usually $\mathbf{C}^5$ will be sufficient.

Suppose (3.1.1) has a fixed point at $(y, \lambda) = (y_0, \lambda_0)$, i.e.,

$$g(y_0, \lambda_0) = 0. \tag{3.1.2}$$

Two questions immediately arise.

1. Is the fixed point stable or unstable?

2. How is the stability or instability affected as $\lambda$ is varied?

To answer Question 1, the first step to take is to examine the linear vector field obtained by linearizing (3.1.1) about the fixed point $(y, \lambda) = (y_0, \lambda_0)$. This linear vector field is given by

$$\dot{\xi} = D_y g(y_0, \lambda_0)\xi, \qquad \xi \in \mathbb{R}^n. \tag{3.1.3}$$

If the fixed point is hyperbolic (i.e., none of the eigenvalues of $D_y g(y_0, \lambda_0)$ lie on the imaginary axis), we know that the stability of $(y_0, \lambda_0)$ in (3.1.1) is determined by the linear equation (3.1.3) (cf. Section 1.1A). This also enables us to answer Question 2, because since hyperbolic fixed points are structurally stable (cf. Section 1.2C), varying $\lambda$ slightly does not change the nature of the stability of the fixed point. This should be clear intuitively, but let us belabor the point slightly.

We know that

$$g(y_0, \lambda_0) = 0, \tag{3.1.4}$$

and that

$$D_y g(y_0, \lambda_0) \tag{3.1.5}$$

has no eigenvalues on the imaginary axis. Therefore, $D_y g(y_0, \lambda_0)$ is invertible. By the implicit function theorem, there thus exists a *unique* $\mathbf{C}^r$ function, $y(\lambda)$, such that

$$g(y(\lambda), \lambda) = 0 \tag{3.1.6}$$

for $\lambda$ sufficiently close to $\lambda_0$ with

$$y(\lambda_0) = y_0. \tag{3.1.7}$$

Now, by continuity of the eigenvalues with respect to parameters, for $\lambda$ sufficiently close to $\lambda_0$,

$$D_y g(y(\lambda), \lambda) \tag{3.1.8}$$

has no eigenvalues on the imaginary axis. Therefore, for $\lambda$ sufficiently close to $\lambda_0$, the hyperbolic fixed point $(y_0, \lambda_0)$ of (3.1.1) persists and its stability type remains unchanged. To summarize, in a neighborhood of $\lambda_0$ an isolated fixed point of (3.1.1) persists and always has the same stability type.

The real fun starts when the fixed point $(y_0, \lambda_0)$ of (3.1.1) is not hyperbolic, i.e., when $D_y g(y_0, \lambda_0)$ has some eigenvalues on the imaginary axis. In this case, for $\lambda$ very close to $\lambda_0$ (and for $y$ close to $y_0$), radically new dynamical behavior can occur. For example, fixed points can be created or destroyed and time-dependent behavior such as periodic, quasiperiodic, or even chaotic dynamics can be created. In a certain sense (to be clarified later), the more eigenvalues on the imaginary axis, the more exotic the dynamics will be.

We will begin our study by considering the simplest way in which $D_y g(y_0, \lambda_0)$ can be nonhyperbolic. This is the case where $D_y g(y_0, \lambda_0)$ has a single zero eigenvalue with the remaining eigenvalues having nonzero real parts. The question we ask in this situation is what is the nature of this nonhyperbolic fixed point for $\lambda$ close to $\lambda_0$? It is under these circumstances where the real power of the center manifold theory becomes apparent, since we know that this question can be answered by studying the vector field (3.1.1) restricted to the associated center manifold (cf. Section 2.1). In this case the vector field on the center manifold will be a $p$-parameter family of one-dimensional vector fields. This represents a vast simplification of (3.1.1).

## 3.1A A Zero Eigenvalue

Suppose that $D_y g(y_0, \lambda_0)$ has a single zero eigenvalue with the remaining eigenvalues having nonzero real parts; then the orbit structure near $(y_0, \lambda_0)$ is determined by the associated center manifold equation, which we write as



$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^p, \tag{3.1.9}$$

where $\mu = \lambda - \lambda_0$. Furthermore, we know that (3.1.9) must satisfy

$$f(0,0) = 0, \tag{3.1.10}$$

$$\frac{\partial f}{\partial x}(0,0) = 0. \tag{3.1.11}$$

Equation (3.1.10) is simply the fixed point condition and (3.1.11) is the zero eigenvalue condition. We remark that (3.1.9) is $\mathbf{C}^r$ if (3.1.1) is $\mathbf{C}^r$. Let us begin by studying a few specific examples. In these examples we will assume

$$\mu \in \mathbb{R}^1.$$

If there are more parameters in the problem (i.e., $\mu \in \mathbb{R}^p$, $p > 1$), we will consider all, except one, as fixed. Later we will consider more carefully the role played by the number of parameters in the problem. We remark also that we have not yet precisely defined what we mean by the term "bifurcation." We will consider this after the following series of examples.

### i) Examples

**Example 3.1.1** Consider the vector field

$$\dot{x} = f(x, \mu) = \mu - x^2, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.1.12}$$

It is easy to verify that

$$f(0,0) = 0 \tag{3.1.13}$$

and

$$\frac{\partial f}{\partial x}(0,0) = 0, \tag{3.1.14}$$

but in this example we can determine much more. The set of all fixed points of (3.1.12) is given by

$$\mu - x^2 = 0$$

or

$$\mu = x^2. \tag{3.1.15}$$

This represents a parabola in the $\mu - x$ plane as shown in Figure 3.1.1. In the figure the arrows along the vertical lines represent the flow generated by (3.1.12) along the $x$-direction. Thus, for $\mu < 0$, (3.1.12) has no fixed points, and the vector field is decreasing in $x$. For $\mu > 0$, (3.1.12) has two fixed points. A simple linear stability analysis shows that one of the fixed points is stable (represented by the solid branch of the parabola), and the other fixed point is unstable (represented by the broken branch of the parabola).

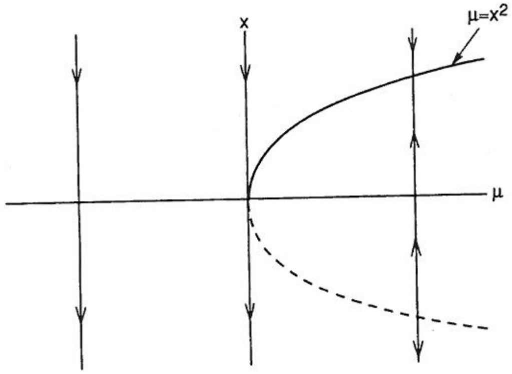


FIGURE 3.1.1.

However, we hope that it is obvious to the reader that, given a $\mathbf{C}^r$ ($r \geq 1$) vector field on $\mathbb{R}^1$ having only two *hyperbolic* fixed points, one must be stable and the other unstable.

This is an example of *bifurcation.* We refer to $(x, \mu) = (0, 0)$ as a *bifurcation point* and the parameter value $\mu = 0$ as a *bifurcation value.*

Figure 3.1.1 is referred to as a *bifurcation diagram.* This particular type of bifurcation (i.e., where on one side of a parameter value there are no fixed points and on the other side there are two fixed points) is referred to as a *saddle-node bifurcation.* Later on we will worry about seeking precise conditions on the vector field on the center manifold that define the saddle-node bifurcation unambiguously.

EXAMPLE 3.1.2 Consider the vector field

$$\dot{x} = f(x, \mu) = \mu x - x^2, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.1.16}$$

It is easy to verify that

$$f(0, 0) = 0 \tag{3.1.17}$$

and

$$\frac{\partial f(0, 0)}{\partial x} = 0. \tag{3.1.18}$$

Moreover, the fixed points of (3.1.16) are given by

$$x = 0 \tag{3.1.19}$$

and

$$x = \mu \tag{3.1.20}$$

and are plotted in Figure 3.1.2. Hence, for $\mu < 0$, there are two fixed points; $x = 0$ is stable and $x = \mu$ is unstable. These two fixed points coalesce at $\mu = 0$ and, for $\mu > 0$, $x = 0$ is unstable and $x = \mu$ is stable. Thus, an



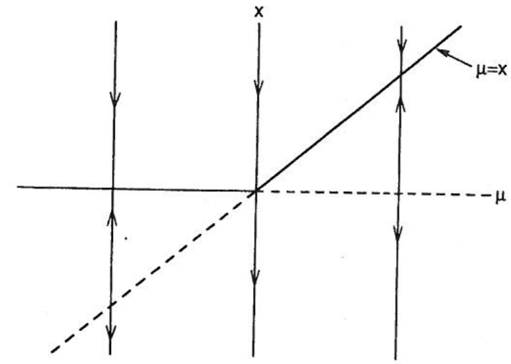


FIGURE 3.1.2.

exchange of stability has occurred at $\mu = 0$. This type of bifurcation is called a *transcritical bifurcation.*

EXAMPLE 3.1.3 Consider the vector field

$$\dot{x} = f(x, \mu) = \mu x - x^3, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.1.21}$$

It is clear that we have

$$f(0, 0) = 0, \tag{3.1.22}$$

$$\frac{\partial f}{\partial x}(0, 0) = 0. \tag{3.1.23}$$

Moreover, the fixed points of (3.1.21) are given by

$$x = 0 \tag{3.1.24}$$

and

$$x^2 = \mu \tag{3.1.25}$$

and are plotted in Figure 3.1.3. Hence, for $\mu < 0$, there is one fixed point, $x = 0$, which is stable. For $\mu > 0$, $x = 0$ is still a fixed point, but two new fixed points have been created at $\mu = 0$ and are given by $x^2 = \mu$. In the process, $x = 0$ has become unstable for $\mu > 0$, with the other two fixed points stable. This type of bifurcation is called a *pitchfork bifurcation.*

EXAMPLE 3.1.4 Consider the vector field

$$\dot{x} = f(x, \mu) = \mu - x^3, \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.1.26}$$

It is trivial to verify that

$$f(0, 0) = 0 \tag{3.1.27}$$

and

$$\frac{\partial f}{\partial x}(0, 0) = 0. \tag{3.1.28}$$

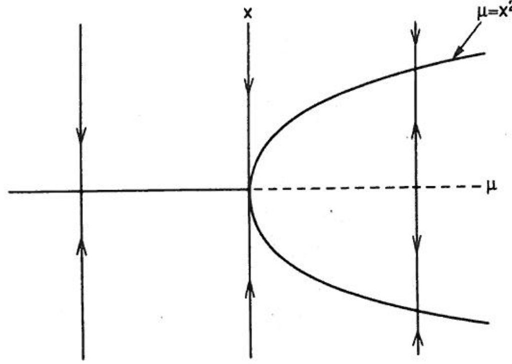


FIGURE 3.1.3.

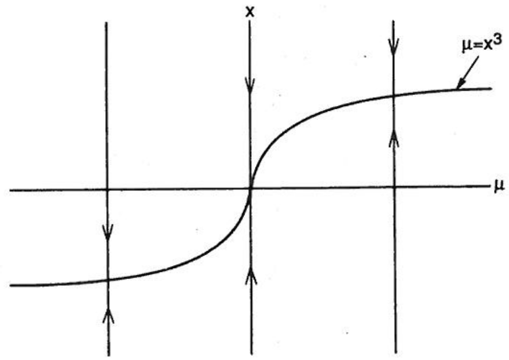


FIGURE 3.1.4.

Moreover, all fixed points of (3.1.26) are given by

$$\mu = x^3 \tag{3.1.29}$$

and are shown in Figure 3.1.4. However in this example, despite (3.1.27) and (3.1.28), the dynamics of (3.1.26) are qualitatively the same for $\mu > 0$ and $\mu < 0$. Namely, (3.1.26) possesses a unique, stable fixed point.

### ii) What Is a "Bifurcation of a Fixed Point"?

The term "bifurcation" is extremely general. We will begin to learn its uses in dynamical systems by understanding its use in describing the orbit structure near nonhyperbolic fixed points. Let us consider what we learned from the previous examples.

In all four examples we had

$$f(0,0) = 0$$

and

$$\frac{\partial f}{\partial x}(0,0) = 0,$$

and yet the orbit structure near $\mu = 0$ was different in all four cases. Hence, knowing that a fixed point has a zero eigenvalue for $\mu = 0$ is not sufficient to determine the orbit structure for $\lambda$ near zero. Let us consider each example individually.

1. (*Example 3.1.1*). In this example a *unique* curve (or branch) of fixed points passed through the orgin. Moreover, the curve lay entirely on one side of $\mu = 0$ in the $\mu - x$ plane.
2. (*Example 3.1.2*). In this example two curves of fixed points intersected at the origin in the $\mu - x$ plane. Both curves existed on either side of $\mu = 0$. However, the stability of the fixed point along a given curve changed on passing through $\mu = 0$.
3. (*Example 3.1.3*). In this example two curves of fixed points intersected at the origin in the $\mu - x$ plane. Only one curve ($x = 0$) existed on both sides of $\mu = 0$; however, its stability changed on passing through $\mu = 0$. The other curve of fixed points lay entirely to one side of $\mu = 0$ and had a stability type that was the opposite of $x = 0$ for $\mu > 0$.
4. (*Example 3.1.4*). This example had a unique curve of fixed points passing through the origin in the $\mu - x$ plane and existing on both sides of $\mu = 0$. Moreover, all fixed points along the curve had the same stability type. Hence, despite the fact that the fixed point $(x, \mu) = (0, 0)$ was nonhyperbolic, the orbit structure was qualitatively the same for all $\mu$.

We want to apply the term "bifurcation" to Examples 3.1.1, 3.1.2, and 3.1.3 but not to Example 3.1.4 to describe the change in orbit structure as $\mu$ passes through zero. We are therefore led to the following definition.

Definition 3.1.1 A fixed point $(x, \mu) = (0, 0)$ of a one-parameter family of one-dimensional vector fields is said to undergo a *bifurcation* at $\mu = 0$ if the flow for $\mu$ near zero and $x$ near zero is *not* qualitatively the same as the flow near $x = 0$ at $\mu = 0$.

Several remarks are now in order concerning this definition.

*Remark 1.* The phrase "qualitatively the same" is a bit vague. It can be made precise by substituting the term "$\mathbf{C}^0$-equivalent" (cf. Section 2.2D), and this is perfectly adequate for the study of the bifurcation of fixed points of *one-dimensional* vector fields. However, we will see that as we explore higher dimensional phase spaces and global bifurcations, how to make mathematically precise the statement "two dynamical systems have qualitatively the same dynamics" becomes more and more ambiguous.

*Remark 2.* Practically speaking, a fixed point $(x_0, \mu_0)$ of a one-dimensional vector field is a bifurcation point if either more than one curve of fixed points passes through $(x_0, \mu_0)$ in the $\mu - x$ plane or if only one curve of fixed points passes $(x_0, \mu_0)$ in the $\mu - x$ plane; then it (locally) lies entirely on one side of the line $\mu = \mu_0$ in the $\mu - x$ plane.

*Remark 3.* It should be clear from Example 3.1.4 that the condition that a fixed point is nonhyperbolic is a necessary but not sufficient condition for bifurcation to occur in one-parameter families of vector fields.

We next turn to deriving general conditions on one-parameter families of one-dimensional vector fields which exhibit bifurcations exactly as in Examples 3.1.1, 3.1.2, and 3.1.3.

### iii) The Saddle-Node Bifurcation

We now want to derive conditions under which a general one-parameter family of one-dimensional vector fields will undergo a saddle-node bifurcation exactly as in Example 3.1.1. These conditions will involve derivatives of the vector field evaluated at the bifurcation point and are obtained by a consideration of the geometry of the curve of fixed points in the $\mu - x$ plane in a neighborhood of the bifurcation point.

Let us recall Example 3.1.1. In this example a *unique* curve of fixed points, parameterized by $x$, passed through $(\mu, x) = (0, 0)$. We denote the curve of fixed points by $\mu(x)$. The curve of fixed points satisfied two properties.

1. It was tangent to the line $\mu = 0$ at $x = 0$, i.e.,
$$\frac{d\mu}{dx}(0) = 0. \tag{3.1.30}$$
2. It lay entirely to one side of $\mu = 0$. Locally, this will be satisfied if we have
$$\frac{d^2\mu}{dx^2}(0) \neq 0. \tag{3.1.31}$$

Now let us consider a general, one-parameter family of one-dimensional vector fields.
$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.1.32}$$
Suppose (3.1.32) has a fixed point at $(x, \mu) = (0, 0)$, i.e.,
$$f(0, 0) = 0. \tag{3.1.33}$$
Furthermore, suppose that the fixed point is not hyperbolic, i.e.,
$$\frac{\partial f}{\partial x}(0, 0) = 0. \tag{3.1.34}$$



Now, if we have
$$\frac{\partial f}{\partial \mu}(0, 0) \neq 0, \tag{3.1.35}$$
then, by the implicit function theorem, there exists a unique function
$$\mu = \mu(x), \qquad \mu(0) = 0 \tag{3.1.36}$$
defined for $x$ sufficiently small such that $f(x, \mu(x)) = 0$. (Note: the reader should check that (3.1.35) holds in Example 3.1.1.) Now we want to derive conditions in terms of derivatives of $f$ evaluated at $(\mu, x) = (0, 0)$ so that we have
$$\frac{d\mu}{dx}(0) = 0, \tag{3.1.37}$$
$$\frac{d^2\mu}{dx^2}(0) \neq 0. \tag{3.1.38}$$
Equations (3.1.37) and (3.1.38), along with (3.1.33), (3.1.34), and (3.1.35), imply that $(\mu, x) = (0, 0)$ is a bifurcation point at which a saddle-node bifurcation occurs.

We can derive expressions for (3.1.37) and (3.1.38) in terms of derivatives of $f$ at the bifurcation point by implicitly differentiating $f$ along the curve of fixed points.

Using (3.1.35), we have
$$f(x, \mu(x)) = 0. \tag{3.1.39}$$
Differentiating (3.1.39) with respect to $x$ gives
$$\frac{df}{dx}(x, \mu(x)) = 0 = \frac{\partial f}{\partial x}(x, \mu(x)) + \frac{\partial f}{\partial \mu}(x, \mu(x))\frac{d\mu}{dx}(x). \tag{3.1.40}$$
Evaluating (3.1.40) at $(\mu, x) = (0, 0)$, we obtain
$$\frac{d\mu}{dx}(0) = \frac{-\dfrac{\partial f}{\partial x}(0, 0)}{\dfrac{\partial f}{\partial \mu}(0, 0)}; \tag{3.1.41}$$
thus we see that (3.1.34) and (3.1.35) imply that
$$\frac{d\mu}{dx}(0) = 0, \tag{3.1.42}$$
i.e., the curve of fixed points is tangent to the line $\mu = 0$ at $x = 0$.

Next, let us differentiate (3.1.40) once more with respect to $x$ to obtain
$$\begin{aligned}\frac{d^2 f}{dx^2}(x, \mu(x)) = 0 = {}& \frac{\partial^2 f}{\partial x^2}(x, \mu(x)) + 2\frac{\partial^2 f}{\partial x \partial \mu}(x, \mu(x))\frac{d\mu}{dx}(x) \\ &+ \frac{\partial^2 f}{\partial \mu^2}(x, \mu(x))\left(\frac{d\mu}{dx}(x)\right)^2 \\ &+ \frac{\partial f}{\partial \mu}(\mu, \mu(x))\frac{d^2\mu}{dx^2}(x).\end{aligned} \tag{3.1.43}$$

Evaluating (3.1.43) at $(\mu, x) = (0,0)$ and using (3.1.41) gives

$$\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{\partial f}{\partial \mu}(0,0)\frac{d^2\mu}{dx^2}(0) = 0$$

or

$$\frac{d^2\mu}{dx^2}(0) = \frac{-\dfrac{\partial^2 f}{\partial x^2}(0,0)}{\dfrac{\partial f}{\partial \mu}(0,0)}. \tag{3.1.44}$$

Hence, (3.1.44) is nonzero provided we have

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0. \tag{3.1.45}$$

Let us summarize. In order for (3.1.32) to undergo a saddle-node bifurcation we must have

$$\left.\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0 \end{aligned}\right\} \quad \text{nonhyperbolic fixed point} \tag{3.1.46}$$

and

$$\frac{\partial f}{\partial \mu}(0,0) \neq 0, \tag{3.1.47}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0. \tag{3.1.48}$$

Equation (3.1.47) implies that a unique curve of fixed points passes through $(\mu, x) = (0,0)$, and (3.1.48) implies that the curve lies locally on one side of $\mu = 0$. It should be clear that the sign of (3.1.44) determines on which side of $\mu = 0$ the curve lies. In Figure 3.1.5 we show both cases without indicating stability and leave it as an exercise for the reader to verify the stability types of the different branches of fixed points emanating from the bifurcation point (see Exercise 3.2).

Let us end our discussion of the saddle-node bifurcation with the following remark. Consider a general one-parameter family of one-dimensional vector fields having a nonhyperbolic fixed point at $(x, \mu) = (0,0)$. The Taylor expansion of this vector field is given as follows

$$f(x,\mu) = a_0\mu + a_1x^2 + a_2\mu x + a_3\mu^2 + \mathcal{O}(3). \tag{3.1.49}$$

Our computations show that the dynamics of (3.1.49) near $(\mu, x) = (0,0)$ are qualitatively the same as one of the following vector fields

$$\dot{x} = \mu \pm x^2. \tag{3.1.50}$$

Hence, (3.1.50) can be viewed as the *normal form* for saddle-node bifurcations.



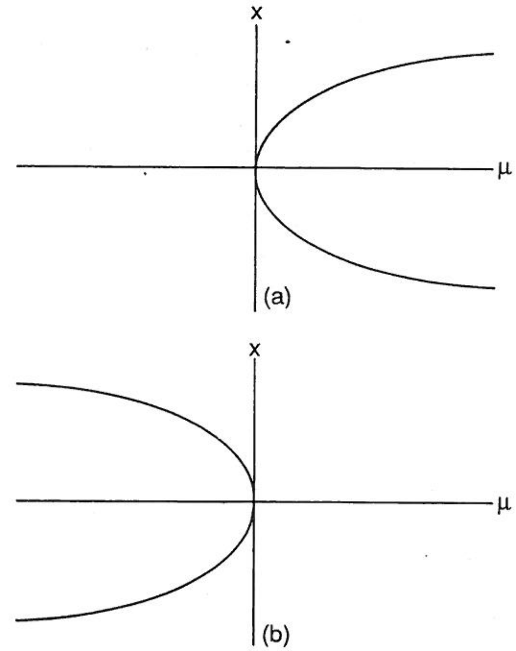


FIGURE 3.1.5. a) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)\right) > 0$; b) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)\right) < 0$.

This brings up another important point. In applying the method of normal forms there is always the question of truncation of the normal form; namely, how are the dynamics of the normal form including only the $\mathcal{O}(k)$ terms modified when the higher order terms are included? We see that, in the study of the saddle-node bifurcation, all terms of $\mathcal{O}(3)$ and higher could be neglected and the dynamics would not be qualitatively changed. The implicit function theorem was the tool that enabled us to verify this fact.

### iv) The Transcritical Bifurcation

We want to follow the same strategy as in our discussion and derivation of general conditions for the saddle-node bifurcation given in the previous section, namely, to use the implicit function theorem to characterize the geometry of the curves of fixed points passing through the bifurcation point in terms of derivatives of the vector field evaluated at the bifurcation point.

For the example of transcritical bifurcation discussed in Example 3.1.2, the orbit structure near the bifurcation point was characterized as follows.

1. Two curves of fixed points passed through $(x, \mu) = (0,0)$, one given by $x = \mu$, the other by $x = 0$.

2. Both curves of fixed points existed on both sides of $\mu = 0$.
3. The stability along each curve of fixed points changed on passing through $\mu = 0$.

Using these three points as a guide, let us consider a general one-parameter family of one-dimensional vector fields

$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1. \tag{3.1.51}$$

We assume that at $(x, \mu) = (0, 0)$, (3.1.51) has a nonhyperbolic fixed point, i.e.,

$$f(0, 0) = 0 \tag{3.1.52}$$

and

$$\frac{\partial f}{\partial x}(0, 0) = 0. \tag{3.1.53}$$

Now, in Example 3.1.2 we had two curves of fixed points passing through $(\mu, x) = (0, 0)$. In order for this to occur it is necessary to have

$$\frac{\partial f}{\partial \mu}(0, 0) = 0, \tag{3.1.54}$$

or else, by the implicit function theorem, only one curve of fixed points could pass through the origin.

Equation (3.1.54) presents a problem if we wish to proceed as in the case of the saddle-node bifurcation; in that situation we used the condition $\frac{\partial f}{\partial \mu}(0, 0) \neq 0$ in order to conclude that a unique curve of fixed points, $\mu(x)$, passed through the bifurcation point. We then evaluated the vector field on the curve of fixed points and used implicit differentiation to derive local characteristics of the geometry of the curve of fixed points based on properties of the derivatives of the vector field evaluated at the bifurcation point. However, if we use Example 3.1.2 as a guide, we can extricate ourselves from this difficulty.

In Example 3.1.2, $x = 0$ was a curve of fixed points passing through the bifurcation point. We will *require* that to be the case for (3.1.51), so that (3.1.51) has the form

$$\dot{x} = f(x, \mu) = xF(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{3.1.55}$$

where, by definition, we have

$$F(x, \mu) \equiv \begin{Bmatrix} \frac{f(x,\mu)}{x}, & x \neq 0 \\ \frac{\partial f}{\partial x}(0, \mu), & x = 0 \end{Bmatrix}. \tag{3.1.56}$$

Since $x = 0$ is a curve of fixed points for (3.1.55), in order to obtain an additional curve of fixed points passing through $(\mu, x) = (0, 0)$ we need to seek conditions on $F$ whereby $F$ has a curve of zeros passing through



$(\mu, x) = (0, 0)$ (that is not given by $x = 0$). These conditions will be in terms of derivatives of $F$ which, using (3.1.56), can be expressed as derivatives of $f$.

Using (3.1.56), it is easy to verify the following

$$F(0, 0) = 0, \tag{3.1.57}$$

$$\frac{\partial F}{\partial x}(0, 0) = \frac{\partial^2 f}{\partial x^2}(0, 0), \tag{3.1.58}$$

$$\frac{\partial^2 F}{\partial x^2}(0, 0) = \frac{\partial^3 f}{\partial x^3}(0, 0), \tag{3.1.59}$$

and (most importantly)

$$\frac{\partial F}{\partial \mu}(0, 0) = \frac{\partial^2 f}{\partial x \partial \mu}(0, 0). \tag{3.1.60}$$

Now let us assume that (3.1.60) is *not* zero; then by the implicit function theorem there exists a function, $\mu(x)$, defined for $x$ sufficiently small, such that

$$F(x, \mu(x)) = 0. \tag{3.1.61}$$

Clearly, $\mu(x)$ is a curve of fixed points of (3.1.55). In order for $\mu(x)$ to not coincide with $x = 0$ and to exist on both sides of $\mu = 0$, we must require that

$$0 < \left| \frac{d\mu}{dx}(0) \right| < \infty.$$

Implicitly differentiating (3.1.61) exactly as in the case of the saddle-node bifurcation we obtain

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial F}{\partial x}(0, 0)}{\frac{\partial F}{\partial \mu}(0, 0)}. \tag{3.1.62}$$

Using (3.1.57), (3.1.58), (3.1.59), and (3.1.60), (3.1.62) becomes

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial^2 f}{\partial x^2}(0, 0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0, 0)}. \tag{3.1.63}$$

We now summarize our results. In order for a vector field

$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{3.1.64}$$

to undergo a transcritical bifurcation, we must have

$$\left. \begin{array}{l} f(0, 0) = 0 \\ \frac{\partial f}{\partial x}(0, 0) = 0 \end{array} \right\} \quad \text{nonhyperbolic fixed point} \tag{3.1.65}$$

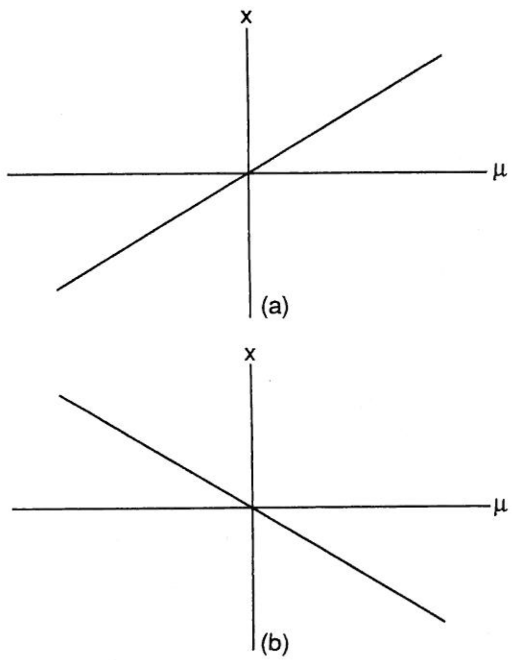


FIGURE 3.1.6. a) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial^2 f}{\partial x\partial\mu}(0,0)\right) > 0$; b) $\left(-\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial^2 f}{\partial x\partial\mu}(0,0)\right) < 0$.

and

$$\frac{\partial f}{\partial \mu}(0,0) = 0, \tag{3.1.66}$$

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0, \tag{3.1.67}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0. \tag{3.1.68}$$

We note that the slope of the curve of fixed points not equal to $x = 0$ is given by (3.1.63). These two cases are shown in Figure 3.1.6; however, we do not indicate stabilities of the different branches of fixed points. We leave it as an exercise to the reader to verify the stability types of the different curves of fixed points emanating from the bifurcation point (see Exercise 3.3).

Thus, (3.1.65), (3.1.66), (3.1.67), and (3.1.68) show that the orbit structure near $(x,\mu) = (0,0)$ is qualitatively the same as the orbit structure near $(x,\mu) = (0,0)$ of

$$\dot{x} = \mu x \mp x^2. \tag{3.1.69}$$

Equation (3.1.69) can be viewed as a normal form for the transcritical bifurcation.



### v) The Pitchfork Bifurcation

The discussion and derivation of conditions under which a general one-parameter family of one-dimensional vector fields will undergo a bifurcation of the type shown in Example 3.1.3 follows very closely our discussion of the transcritical bifurcation.

The geometry of the curves of fixed points associated with the bifurcation in Example 3.1.3 had the following characteristics.

1. Two curves of fixed points passed through $(\mu, x) = (0,0)$, one given by $x = 0$, the other by $\mu = x^2$.
2. The curve $x = 0$ existed on both sides of $\mu = 0$; the curve $\mu = x^2$ existed on one side of $\mu = 0$.
3. The fixed points on the curve $x = 0$ had different stability types on opposite sides of $\mu = 0$. The fixed points on $\mu = x^2$ all had the same stability type.

Now we want to consider conditions on a general one-parameter family of one-dimensional vector fields having two curves of fixed points passing through the bifurcation point in the $\mu - x$ plane that have the properties given above.

We denote the vector field by

$$\dot{x} = f(x,\mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{3.1.70}$$

and we suppose

$$f(0,0) = 0, \tag{3.1.71}$$

$$\frac{\partial f}{\partial x}(0,0) = 0. \tag{3.1.72}$$

As in the case of the transcritical bifurcation, in order to have more than one curve of fixed points passing through $(\mu, x) = (0,0)$ we must have

$$\frac{\partial f}{\partial \mu}(0,0) = 0. \tag{3.1.73}$$

Proceeding further along these lines, we *require* $x = 0$ to be a curve of fixed points for (3.1.70) by assuming the vector field (3.1.70) has the form

$$\dot{x} = xF(x,\mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{3.1.74}$$

where

$$F(x,\mu) \equiv \left\{ \begin{array}{ll} \frac{f(x,\mu)}{x}, & x \neq 0 \\ \frac{\partial f}{\partial x}(0,\mu), & x = 0 \end{array} \right\}. \tag{3.1.75}$$

In order to have a second curve of fixed points passing through $(\mu, x) = (0,0)$ we must have

$$F(0,0) = 0 \tag{3.1.76}$$

with

$$\frac{\partial F}{\partial \mu}(0,0) \neq 0. \tag{3.1.77}$$

Equation (3.1.77) insures that only *one* additional curve of fixed points passes through $(\mu, x) = (0,0)$. Also, using (3.1.77), the implicit function theorem implies that for $x$ sufficiently small there exists a unique function $\mu(x)$ such that

$$F(x, \mu(x)) = 0. \tag{3.1.78}$$

In order for the curve of fixed points, $\mu(x)$, to satisfy the above-mentioned characteristics, it is sufficient to have

$$\frac{d\mu}{dx}(0) = 0 \tag{3.1.79}$$

and

$$\frac{d^2\mu}{dx^2}(0) \neq 0. \tag{3.1.80}$$

The conditions for (3.1.79) and (3.1.80) to hold in terms of the derivatives of $F$ evaluated at the bifurcation point can be obtained via implicit differentiation of (3.1.78) along the curve of fixed points exactly as in the case of the saddle-node bifurcation. They are given by

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)} = 0 \tag{3.1.81}$$

and

$$\frac{d^2\mu}{dx^2}(0) = \frac{-\frac{\partial^2 F}{\partial x^2}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)} \neq 0. \tag{3.1.82}$$

Using (3.1.75), (3.1.81) and (3.1.82) can be expressed in terms of derivatives of $f$ as follows

$$\frac{d\mu}{dx}(0) = \frac{-\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)} = 0 \tag{3.1.83}$$

and

$$\frac{d^2\mu}{dx^2}(0) = \frac{-\frac{\partial^3 f}{\partial x^3}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)} \neq 0. \tag{3.1.84}$$

We summarize as follows. In order for the vector field

$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \quad \mu \in \mathbb{R}^1, \tag{3.1.85}$$

to undergo a pitchfork bifurcation at $(x, \mu) = (0,0)$, it is sufficient to have

$$\left.\begin{array}{l} f(0,0) = 0 \\ \frac{\partial f}{\partial x}(0,0) = 0 \end{array}\right\} \quad \text{nonhyperbolic fixed point} \tag{3.1.86}$$



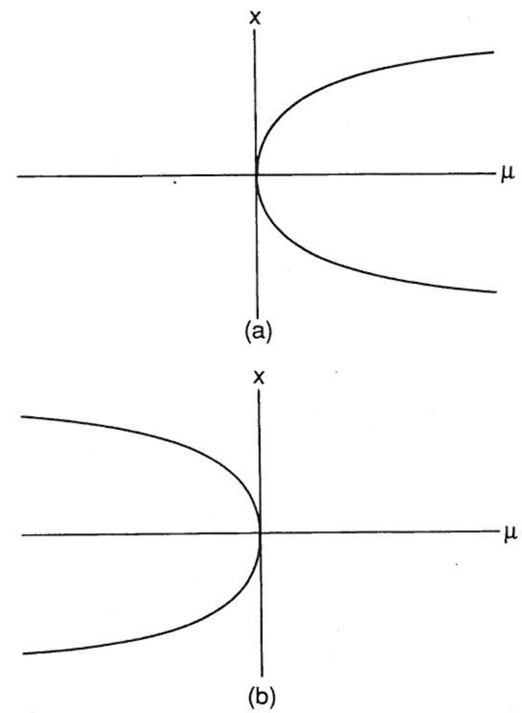


FIGURE 3.1.7. a) $\left(-\frac{\partial^3 f}{\partial x^3}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) > 0$; b) $\left(-\frac{\partial^3 f}{\partial x^3}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)\right) < 0$.

with

$$\frac{\partial f}{\partial \mu}(0,0) = 0, \tag{3.1.87}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0, \tag{3.1.88}$$

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0, \tag{3.1.89}$$

$$\frac{\partial^3 f}{\partial x^3}(0,0) \neq 0. \tag{3.1.90}$$

There are two possibilities for the disposition of the two branches of fixed points depending on the sign of (3.1.84). These two possibilities are shown in Figure 3.1.7 without indicating stabilities. We leave it as an exercise for the reader to verify the stability types for the different branches of fixed points emanating from the bifurcation point (see Exercise 3.4).

We conclude by noting that (3.1.86), (3.1.87), (3.1.88), (3.1.89), and (3.1.90) imply that the orbit structure near $(x, \mu) = (0,0)$ is qualitatively the same as the orbit structure near $(x, \mu) = (0,0)$ in the vector field

$$\dot{x} = \mu x \mp x^3. \tag{3.1.91}$$