Theorem 1. Let $X_1, \ldots, X_k, Y$ be Banach spaces. For each $i = 1, \ldots, k$, let $D_i$ be an open subset of $X_i$. Let $f : D_1 \times \cdots \times D_k \to Y$ be a function. Assume:

(1) At each $(x_1, \ldots, x_k) \in D_1 \times \cdots \times D_k$, each partial derivative $D_i f(x_1, \ldots, x_k)$ exists.

(2) For each $i$, the mapping from $D_1 \times \cdots \times D_k$ to $L(X_i, Y)$ given by $(x_1, \ldots, x_k) \to D_i f(x_1, \ldots, x_k)$ is continuous.

Then $f$ is $C^1$, and $D f(x_1, \ldots, x_k)(h_1, \ldots, h_k) = D_1 f(x_1, \ldots, x_k) h_1 + \cdots + D_k f(x_1, \ldots, x_k) h_k$.

Proof. We’ll do the case $k = 2$ only. Differentiability:

$$f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) - (D_1 f(x_1, x_2) h_1 + D_2 f(x_1, x_2) h_2)$$

$$= (\int_0^1 D_2 f(x_1 + h_1, x_2 + sh_2) - D_2 f(x_1, x_2) \, ds) h_2$$

$$+ (\int_0^1 D_1 f(x_1 + sh_1, x_2) - D_1 f(x_1, x_2) \, ds) h_1.$$ 

Since each $D_i f(x_1, x_2)$ depends continuously on $(x_1, x_2)$, for $\epsilon > 0$ we have

$$|f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) - (D_1 f(x_1, x_2) h_1 + D_2 f(x_1, x_2) h_2)|$$

$$\leq \sup_{0 \leq s \leq 1} |D_2 f(x_1 + h_1, x_2 + sh_2) - D_2 f(x_1, x_2)| |h_2|$$

$$+ \sup_{0 \leq s \leq 1} |D_1 f(x_1 + sh_1, x_2) - D_1 f(x_1, x_2)| |h_1| \leq \frac{\epsilon}{2} |h_2| + \frac{\epsilon}{2} |h_1| \leq \epsilon |(h_1, h_2)|$$

for $|(h_1, h_2)|$ sufficiently small. This shows that $f$ is differentiable and the derivative is the given formula.

$C^1$:

$$(D f(x'_1, x'_2) - D f(x_1, x_2))(h_1, h_2)$$

$$= (D_1 f(x'_1, x'_2) - D_1 f(x_1, x_2)) h_1 + (D_2 f(x'_1, x'_2) - D_2 f(x_1, x_2)) h_2.$$ 

Since each $D_i f(x_1, x_2)$ depends continuously on $(x_1, x_2)$, for $\epsilon > 0$ we have

$$| (D f(x'_1, x'_2) - D f(x_1, x_2))(h_1, h_2) | \leq \frac{\epsilon}{2} |h_1| + \frac{\epsilon}{2} |h_2| \leq \epsilon |(h_1, h_2)|$$

for $|(x'_1, x'_2) - (x_1, x_2)|$ sufficiently small. Hence for $|(x'_1, x'_2) - (x_1, x_2)|$ sufficiently small, $\|D f(x'_1, x'_2) - D f(x_1, x_2)\| \leq \epsilon$. \hfill \Box
Theorem 2. Let $X_1, \ldots, X_k, Y$ be Banach spaces. Let $M : X_1 \times \cdots \times X_k \to Y$ be a bounded $k$-multilinear map. Then $M$ is $C^1$, and $DM(x_1, \ldots, x_k)(h_1, \ldots, h_k) = M(h_1, x_2, \ldots, x_k) + M(x_1, h_2, x_3, \ldots, x_k) + \cdots + M(x_1, \ldots, x_{k-1}, h_k)$.

Proof. The steps are:

1. If we fix everything but $x_i$, then $M$ is linear in $x_i$. Therefore
   \[
   D_iM(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k)h_i = M(x_1, \ldots, x_{i-1}, h_i, x_{i+1}, \ldots, x_k).
   \]
   This is a bounded linear map because
   \[
   |M(x_1, \ldots, x_{i-1}, h_i, x_{i+1}, \ldots, x_k)| \leq \|M\| |x_1| \cdots |x_{i-1}| |h_i| |x_{i+1}| \cdots |x_k|,
   \]
   so a bound is $\|M\| |x_1| \cdots |x_{i-1}| |x_{i+1}| \cdots |x_k|$.

2. Now we just need to show that for each $i$, the map $(x_1, \ldots, x_k) \to D_i f(x_1, \ldots, x_k)$ is continuous. Then we can apply the previous theorem to get the result.
   Let’s just look at the case $k = i = 3$. We want to show that the $(x_1, x_2, x_3) \to D_3 f(x_1, x_2, x_3)$ is continuous. The map $(x_1, x_2, x_3) \to D_3 f(x_1, x_2, x_3)$, from $X_1 \times X_2 \times X_3$ to $L(X_3, Y)$, takes $(x_1, x_2, x_3)$ to the linear map $h_3 \to M(x_1, x_2, h_3)$. It is the composition of two maps:
   (a) $(x_1, x_2, x_3) \to (x_1, x_2)$ from $X_1 \times X_2 \times X_3$ to $X_1 \times X_2$.
   (b) $(x_1, x_2) \to$ the linear map $h_3 \to M(x_1, x_2, h_3)$, from $X_1 \times X_2$ to $L(X_3, Y)$.
   The first is bounded linear, the second is bounded bilinear. Thus each is continuous, so the composition is continuous.

$\square$