TWO MORE THEOREMS ABOUT DIFFERENTIATION IN BANACH SPACE

S. SCHECTER

Theorem 1. *Let $X_1, \dots, X_k, Y$ be Banach spaces. For each $i = 1, \dots, k$, let D_i be an open subset of X_i. Let $f : D_1 \times \cdots \times D_k \to Y$ be a function. Assume:*

1. *At each $(x_1, \dots, x_k) \in D_1 \times \cdots \times D_k$, each partial derivative $D_i f(x_1, \dots, x_k)$ exists.*
2. *For each i, the mapping from $D_1 \times \cdots \times D_k$ to $L(X_i, Y)$ given by $(x_1, \dots, x_k) \to D_i f(x_1, \dots, x_k)$ is continuous.*

Then f is C^1, and $Df(x_1, \dots, x_k)(h_1, \dots, h_k) = D_1 f(x_1, \dots, x_k)h_1 + \cdots + D_k f(x_1, \dots, x_k)h_k$.

Proof. We'll do the case $k = 2$ only. Differentiability:

$$\begin{aligned}
&f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) - (D_1 f(x_1, x_2)h_1 + D_2 f(x_1, x_2)h_2) \\
&= f(x_1+h_1, x_2+h_2) - f(x_1+h_1, x_2) + f(x_1+h_1, x_2) - f(x_1, x_2) - (D_1 f(x_1, x_2)h_1 + D_2 f(x_1, x_2)h_2) \\
&= \left(\int_0^1 D_2 f(x_1 + h_1, x_2 + sh_2) - D_2 f(x_1, x_2)\, ds \right) h_2 \\
&\quad + \left(\int_0^1 D_1 f(x_1 + sh_1, x_2) - D_1 f(x_1, x_2)\, ds \right) h_1.
\end{aligned}$$

Since each $D_i f(x_1, x_2)$ depends continuously on (x_1, x_2), for $\epsilon > 0$ we have

$$\begin{aligned}
&|f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) - (D_1 f(x_1, x_2)h_1 + D_2 f(x_1, x_2)h_2)| \\
&\le \sup_{0 \le s \le 1} |D_2 f(x_1 + h_1, x_2 + sh_2) - D_2 f(x_1, x_2)|\, |h_2| \\
&+ \sup_{0 \le s \le 1} |D_1 f(x_1 + sh_1, x_2) - D_1 f(x_1, x_2)|\, |h_1| \le \frac{\epsilon}{2}|h_2| + \frac{\epsilon}{2}|h_1| \le \epsilon |(h_1, h_2)|
\end{aligned}$$

for $|(h_1, h_2)|$ sufficiently small. This shows that f is differentiable and the derivative is the given formula.

C^1:

$$\begin{aligned}
&(Df(x_1', x_2') - Df(x_1, x_2))(h_1, h_2) \\
&= (D_1 f(x_1', x_2') - D_1 f(x_1, x_2))\, h_1 + (D_2 f(x_1', x_2') - D_2 f(x_1, x_2))\, h_2.
\end{aligned}$$

Since each $D_i f(x_1, x_2)$ depends continuously on (x_1, x_2), for $\epsilon > 0$ we have

$$|(Df(x_1', x_2') - Df(x_1, x_2))(h_1, h_2)| \le \frac{\epsilon}{2}|h_1| + \frac{\epsilon}{2}|h_2| \le \epsilon |(h_1, h_2)|$$

for $|(x_1', x_2') - (x_1, x_2)|$ sufficiently small. Hence for $|(x_1', x_2') - (x_1, x_2)|$ sufficiently small, $\|Df(x_1', x_2') - Df(x_1, x_2)\| \le \epsilon$. □

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Theorem 2. *Let $X_1, \ldots, X_k, Y$ be Banach spaces. Let $M : X_1 \times \cdots \times X_k \to Y$ be a bounded k-multilinear map. Then M is C^1, and $DM(x_1, \ldots, x_k)(h_1, \ldots, h_k) = M(h_1, x_2, \ldots, x_k) + M(x_1, h_2, x_3, \ldots, x_k) + \cdots + M(x_1, \ldots, x_{k-1}, h_k)$.*

Proof. The steps are:

(1) If we fix everything but x_i, then M is linear in x_i. Therefore

$$D_i M(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k)h_i = M(x_1, \ldots, x_{i-1}, h_i, x_{i+1}, \ldots, x_k).$$

This is a bounded linear map because

$$|M(x_1, \ldots, x_{i-1}, h_i, x_{i+1}, \ldots, x_k)| \leq \|M\||x_1| \cdots |x_{i-1}||h_i||x_{i+1}| \cdots |x_k|,$$

so a bound is $\|M\||x_1| \cdots |x_{i-1}||x_{i+1}| \cdots |x_k|$.

(2) Now we just need to show that for each i, the map $(x_1, \ldots, x_k) \to D_i f(x_1, \ldots, x_k)$ is continuous. Then we can apply the previous theorem to get the result.

Let's just look at the case $k = i = 3$. We want to show that the $(x_1, x_2, x_3) \to D_3 f(x_1, x_2, x_3)$ is continuous. The map $(x_1, x_2, x_3) \to D_3 f(x_1, x_2, x_3)$, from $X_1 \times X_2 \times X_3$ to $L(X_3, Y)$, takes (x_1, x_2, x_3) to the linear map $h_3 \to M(x_1, x_2, h_3)$. It is the composition of two maps:

(a) $(x_1, x_2, x_3) \to (x_1, x_2)$ from $X_1 \times X_2 \times X_3$ to $X_1 \times X_2$.

(b) $(x_1, x_2) \to$ the linear map $h_3 \to M(x_1, x_2, h_3)$, from $X_1 \times X_2$ to $L(X_3, Y)$.

The first is bounded linear, the second is bounded bilinear. Thus each is continuous, so the composition is continuous.

□