TWO THEOREMS ABOUT DIFFERENTIATION IN BANACH SPACE

S. SCHECTER

Theorem 1. *Suppose X is a Banach space, $H : [a, b] \to X$, and $H'(t) = 0$ for all $t \in [a, b]$. Then H is constant.*

Proof. Let $\epsilon > 0$. Let $I_\epsilon = \{t \in [a, b] : |H(t) - H(a)| \leq \epsilon(t - a)\}$. We will show that $I_\epsilon = [a, b]$. Once this is done, let $t \in (a, b]$. Then for every $\epsilon > 0$, $t \in I_\epsilon$. It follows easily that $|H(t) - H(a)| = 0$, so $H(t) = H(a)$.

To show that $I_\epsilon = [a, b]$, we use the following steps.

1. Since H is differentiable, H is continuous. (An easy-to-prove lemma.) Let $J_\epsilon = [a, b]\backslash I_\epsilon$. Since H is continuous, J_ϵ is open in $[a, b]$. (Easy to show.)

2. Assume J_ϵ is nonempty and derive a contradiction. This completes the proof.

To do step 2, assume that J_ϵ is nonempty. Let $c = \inf J_\epsilon$.

Since J_ϵ is open, we cannot have $c = b$. So assume $c < b$.

Notice :

(1) If $c = a$, then $c \in I_\epsilon$ (because it is obvious that $a \in I_\epsilon$).
(2) If $c > a$, we cannot have $c \in J_\epsilon$ because J_ϵ is open in $[a, b]$, so $c \in I_\epsilon$.

We conclude that $c \in I_\epsilon$.

Since $H'(c) = 0$, there is a number $\delta > 0$ such that if $c < t < c + \delta$, then $|H(t) - H(c)| < \epsilon(t - c)$. For such t,

$$|H(t) - H(a)| \leq |H(t) - H(c)| + |H(c) - H(a)| < \epsilon(t - c) + \epsilon(c - a) = \epsilon(t - a).$$

Hence if $c \leq t < c + \delta$, then $c \in I_\epsilon$. This contradicts our assumption that $c = \inf J_\epsilon$. $\square$

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be C^1. Let J be an interval. Define*

$$N : C^0(J, \mathbb{R}^n) \to C^0(J, \mathbb{R}^m)$$

as follows: if $\phi \in C^0(J, \mathbb{R}^n)$, then $N(\phi)(t) = (f \circ \phi)(t)$. Then N is C^1.

Proof. We use the following steps:

(1) For each $\phi \in C^0(J, \mathbb{R}^n)$ define a map $B : C^0(J, \mathbb{R}^n) \to C^0(J, \mathbb{R}^m)$ that we believe to be $DN(\phi)$. Since B will depend on ϕ, we denote it $B(\phi)$.
(2) Show that for each ϕ, $B(\phi)$ is actually a linear map from $C^0(J, \mathbb{R}^n)$ to $C^0(J, \mathbb{R}^m)$.
(3) Show that for each ϕ, $B(\phi)$ is actually a *bounded* linear map from $C^0(J, \mathbb{R}^n)$ to $C^0(J, \mathbb{R}^m)$. Thus $B(\phi) \in L(C^0(J, \mathbb{R}^n), C^0(J, \mathbb{R}^m))$.
(4) Show that $DN(\phi) = B(\phi)$.
(5) Show that the mapping $B : C^0(J, \mathbb{R}^n) \to L(C^0(J, \mathbb{R}^n), C^0(J, \mathbb{R}^m))$ is continuous.

Here are the steps:

1. For $\psi \in C^0(J, \mathbb{R}^n)$, we define $B(\phi)\psi$ to be the element of $C^0(J, \mathbb{R}^m)$ given by

$$B(\phi)\psi(t) = Df(\phi(t))\psi(t).$$

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2. $B(\phi)$ is linear: look at one t at a time, then interpret.

$$B(\phi)(\psi_1+\psi_2)(t) = Df(\phi(t))(\psi_1+\psi_2)(t) = Df(\phi(t))(\psi_1(t)+\psi_2(t))$$
$$= Df(\phi(t))\psi_1(t) + Df(\phi(t))\psi_2(t) = B(\phi)\psi_1(t) + B(\phi)\psi_2(t).$$

Therefore $B(\phi)(\psi_1+\psi_2) = B(\phi)\psi_1 + B(\phi)\psi_2$. The proof that $B(\phi)(a\psi) = aB(\phi)\psi$ is similar.

3. $B(\phi)$ is bounded linear: look at one t at a time, but some work is needed to get an estimate that works for all $t \in J$.

$$|B(\phi)\psi(t)| = |Df(\phi(t))\psi(t)| \le \|Df(\phi(t))\||\psi(t)| \le \|Df(\phi(t))\||\psi||.$$

Since $\phi \in C^0(J,\mathbb{R}^n)$, ϕ is a *bounded* continuous map (this is part of the definition of $C^0(J,\mathbb{R}^n)$). Thus $\{\phi(t) : t \in J\}$ is a bounded subset of $\mathbb{R}^n$. Since $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1, $\|Df(x)\|$ is bounded on any bounded subset of $\mathbb{R}^n$. Therefore there is a number $K > 0$ such that $\|Df(\phi(t))\| \le K$ for all $t \in J$, so

$$|B(\phi)\psi(t)| \le K|\psi|.$$

Now $|B(\phi)\psi|$ is just the sup over $t \in J$ of the left hand side. Since we have the same estimate for every t, $|B(\phi)\psi| \le K|\psi|$. Since the same K works for any ψ (look at how we chose it), $B(\phi)$ is a bounded linear map.

4. $DN(\phi) = B(\phi)$: start by looking at one t at a time.

$$\big(N(\phi+\psi) - N(\phi) - B(\phi)\psi\big)(t) = f(\phi(t)+\psi(t)) - f(\phi(t)) - Df(\phi(t))\psi(t)$$
$$= \int_0^1 \Big(Df(\phi(t)+s\psi(t)) - Df(\phi(t))\Big)\psi(t)\,ds.$$

Let E be a compact subset of $\mathbb{R}^n$ that includes the closed ball of radius 1 about every point $\phi(t)$, $t \in J$. (There is such a set E because, as you will recall, $\{\phi(t) : t \in J\}$ is a bounded subset of $\mathbb{R}^n$.) The map $x \to Df(x)$ is continuous on E; since E is compact, it is uniformly continuous. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that if x_1, $x_2 \in E$ and $|x_2 - x_1| < \delta$, then $\|Df(x_2) - Df(x_1)\| < \epsilon$. We may assume $\delta < 1$. Then if $\psi \in C^0(J,\mathbb{R}^n)$ with $|\psi| < \delta$ and $t \in J$, we have $\|Df(\phi(t)+s\psi(t)) - Df(\phi(t))\| < \epsilon$. Therefore

$$\big|\big(N(\phi+\psi) - N(\phi) - B(\phi)\psi\big)(t)\big| = \left|\int_0^1 \Big(Df(\phi(t)+s\psi(t)) - Df(\phi(t))\Big)\psi(t)\,ds\right|$$
$$\le \epsilon|\psi(t)| \le \epsilon|\psi|.$$

Taking the sup over $t \in J$ of the left hand side, we get $\big|N(\phi+\psi) - N(\phi) - B(\phi)\psi\big| \le \epsilon|\psi|$ when $|\psi| < \delta$. Since ϵ was arbitrary, this proves that $DN(\phi) = B(\phi)$.

5. The mapping $B : C^0(J,\mathbb{R}^n) \to L(C^0(J,\mathbb{R}^n), C^0(J,\mathbb{R}^m))$ is continuous: Let $\phi_1 \in C^0(J,\mathbb{R}^n)$. We'll show that B is continuous at ϕ_1. Note that

$$(B(\phi_2) - B(\phi_1))\psi(t) = \Big(Df(\phi_2(t)) - Df(\phi_1(t))\Big)\psi(t).$$

Let E be a compact subset of $\mathbb{R}^n$ that includes the closed ball of radius 1 about every point $\phi_1(t)$, $t \in J$. Let $\epsilon > 0$. As in step 4, there is a number δ, $0 < \delta < 1$, such that if x_1, $x_2 \in E$ and $|x_2 - x_1| < \delta$, then $\|Df(x_2) - Df(x_1)\| < \epsilon$. Then if $|\phi_2 - \phi_1| < \delta$,

$$|(B(\phi_2) - B(\phi_1))\psi(t)| \le \|Df(\phi_2(t)) - Df(\phi_1(t))\||\psi(t)| \le \epsilon|\psi|.$$

Taking the sup over $t \in J$ of the left hand side, we get $|\big(B(\phi_2) - B(\phi_1)\big)\psi| \leq \epsilon|\psi|$. Therefore $\|B(\phi_2) - B(\phi_1)\| \leq \epsilon$ when $|\phi_2 - \phi_1| < \delta$. Since ϵ was arbitrary, this shows that B is continuous at ϕ_1. $\square$