MA 732 Homework 9

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1. Consider the differential equation

$$\dot{x} = x^2 + y^2 - 1,$$
$$\dot{y} = 5(xy - 1).$$

   (a) Show that there are no equilibria.

   (b) Are there any closed orbits? Explain briefly.

   (c) Use the method of nullclines to draw the phase portrait in the finite plane. (Draw the curves where $\dot{x} = 0$ and $\dot{y} = 0$. Determine the signs of $\dot{x}$ and $\dot{y}$ in the regions between these curves. Use this information to draw some typical solutions.)

   (d) Use the coordinates $x = \frac{1}{\rho}\cos\theta$, $y = \frac{1}{\rho}\sin\theta$ to draw the phase portrait near the "circle at infinity" $\rho = 0$. Hint: there are six equilibria on $\rho = 0$.

   (e) The work you have done so far suggests that if $(x(t), y(t))$ is any solution, then as t increases, $\frac{y(t)}{x(t)}$ approaches -2, 0, or 2, and as t decreases, $\frac{y(t)}{x(t)}$ approaches -2, 0, or 2. Assume this, and suppose $(x(0), y(0))$ lies in the region $\{(x, y) : x > 0 \text{ and } xy > 1\}$. Show that as t increases, $\frac{y(t)}{x(t)} \to 2$.

   (f) Extra credit: try to *prove* that if $(x(t), y(t))$ is any solution, then as t increases, $\frac{y(t)}{x(t)}$ approaches -2, 0, or 2, and as t decreases, $\frac{y(t)}{x(t)}$ approaches -2, 0, or 2.

2. Let $\dot{x} = f(x, \mu) = xa(x, \mu)$, where $a(x, \mu)$ is C^1, $a(0, 0) = 0$, $a_x(0, 0) = H \neq 0$, and $a_\mu(0, 0) = I \neq 0$.

   (a) Prove: There is a neighborhood of $(0, 0)$ in $x\mu$-space in which all solutions of $f(x, \mu) = 0$ lie on two curves, the first given by $x = 0$, the second given by $x = k(\mu)$, where k is C^1, $k(0) = 0$, and $k'(0) = -\frac{I}{H}$.

   (b) Let $g(\mu) = f_x(0, \mu)$. Show that $g(0) = 0$ and $g'(0) = I$. What does this tell us about which of the equilibria $(0, \mu)$ near the origin are attractors and which are repellers?

   (c) Let $h(\mu) = f_x(k(\mu), \mu)$ Show that $h(0) = 0$ and $h'(0) = -I$. What does this tell us about which of the equilibria $(k(\mu), \mu)$ near the origin are attractors and which are repellers?

3. Use center manifold reduction to show that

$$\begin{aligned}\dot{x} &= y - 2x, \\ \dot{y} &= \mu + x^2 - y\end{aligned}$$

has a saddle-node bifurcation at $(x, y, \mu) = (1, 2, 1)$. (Suggestion: Shift coordinates to put this point at $(0, 0, 0)$.) Draw the phase portrait on the center manifold near this point, and explain how your picture is related to the full phase portrait near this point.