MA 732 Homework 8

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1. Let $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ be a C^1 differential equation in the plane. Assume (1) $(0,0)$ is an equilibrium; (2) both eigenvalues of the linearization at $(0,0)$ have positive real part; (3) there is a circle $r = r_0$, $r_0 > 0$, on which $\dot{r} < 0$; (4) there are no equilibria with $0 < r < r_0$. Use the Poincaré-Bendixson Theorem to show that there is a closed orbit in the region $0 < r < r_0$.

2. Use the formula

$$I_f(L) = \frac{1}{2\pi}\int_L \frac{PdQ - QdP}{P^2 + Q^2}$$

(text, p. 214) to compute the index of the following vector fields around a positively-oriented loop that surrounds the origin. (For example, you can use the loop given by $(x(s), y(s)) = (\cos s, \sin s)$, $0 \le s \le 2\pi$.)

   (a) $\dot{x} = ax$, $\dot{y} = by$, with $a > 0$ and $b > 0$.

   (b) $\dot{x} = -ax$, $\dot{y} = by$, with $a > 0$ and $b > 0$.

3. Consider the differential equation $\dot{x} = y + x - \frac{x^3}{3}$, $\dot{y} = -x$.

   (a) Draw the nullclines and the vector field on the nullclines. Use this information to sketch the approximate direction of the vector field in the open regions determined by the nullclines.

   (b) From the above information alone, does it seem possible that there is a closed orbit that lies between the lines $x = -1$ and $x = 1$?

   (c) Use Bendixson's Criterion to show that there is no closed orbit that lies between the lines $x = -1$ and $x = 1$.

   (d) Let $(x(t), y(t))$ be a solution of the differential equation. Let $(\tilde{x}(t), \tilde{y}(t)) = (-x(t), -y(t))$. Show that $(\tilde{x}(t), \tilde{y}(t))$ is also a solution.

   (e) Let $(x(t), y(t))$, $0 \le t \le T$ be a solution with $(x(0), y(0)) = (0, a)$, $a > 0$; $(x(T), y(T)) = (0, -b)$, $b > 0$; and $x(t) > 0$ for $0 < t < T$. Show: if $a \ne b$, then the curve $(x(t), y(t))$, $0 \le t \le T$, is not part of a closed orbit. Suggestion: Just treat one of the cases $a < b$ or $a > b$. Draw the curves $(x(t), y(t))$, $0 \le t \le T$, and $(\tilde{x}(t), \tilde{y}(t))$, $0 \le t \le T$ (see part (d)), and draw your conclusions.

   (f) Use parts (c) and (e) to show that any closed orbit must cross both the line $x = -1$ and the line $x = 1$.