MA 732 Homework 6 Revised

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1. Consider the differential equation (5.42) in Meiss, p. 193. We are interested in solutions that start in the open positive orthant, i.e., $x(0)$, $y(0)$, and $z(0)$ are all positive. Note that the planes $x = 0$, $y = 0$, and $z = 0$ are all invariant. Thus a solution that starts in the open positive orthant stays in it for all time.

   (a) Let $R(x, y, z) = x + y + z$. Show that $\dot{R} = R - R^2$. Use the phase portrait for this equation to explain why any solution of (5.42) that starts in the positive orthant approaches the plane $R = 1$ as t increases.

   (b) Since DR is surjective and $R = 1$ implies $\dot{R} = 0$, the plane $R = 1$ is invariant for (5.42). Using x and y as the coordinates on this plane, find the restriction of the differential equation to this plane. (You will have $\dot{x}$ and $\dot{y}$ in terms of x and y only.) Answer:

$$\dot{x} = f(x, y) = \delta x(1 - x - 2y), \tag{1}$$
$$\dot{y} = g(x, y) = \delta y(-1 + 2x + y). \tag{2}$$

   (c) Find the equilibria of (1)–(2).

   (d) Show that $f_x = (-g)_y$.

   (e) Part (d) implies that there is a function $H(x, y)$ such that $H_y = f$ and $H_x = -g$. Find this function. (This sort of thing is down in third-semester calculus. Answer: $H(x, y) = \delta xy(1 - x - y)$.) This shows that the system (1)–(2) is Hamiltonian, with the Hamiltonian being $H(x, y)$. (See the top of p. 128.)

   (f) It follows that solutions of (1)–(2) are contained in level curves of H. Note that the "level curve" $H = 0$ is actually the union of the lines $x = 0$, $y = 0$, and $x + y = 1$. They make a triangle. What do the level curves inside this triangle look like? You can use a program like Maple to answer this if you want. (You will need to choose a value for δ in order to use Maple. The Maple command contourplot should help.)

   (g) Describe the behavior of a typical solution of (5.42) in the open positive orthant as t increases.

2. Consider the differential equation

$$\dot{x} = xy, \tag{3}$$
$$\dot{y} = x^2 + 2y^2 + x^3. \tag{4}$$

Use polar coordinates to draw the phase portrait of this differential equation near the origin.

3. Use center manifold reduction to describe the phase portrait near the origin for the system

$$\begin{aligned}\dot{x} &= -y + xz,\\ \dot{y} &= x + yz,\\ \dot{z} &= -z - x^2 - y^2 + z^2.\end{aligned}$$

Suggestion: The eigenvalues are $\pm i$ and -1. The center subspace is the xy-plane. The center manifold is therefore

$$z = h(x, y) = 0 + 0 \cdot x + 0 \cdot y + Ax^2 + Bxy + Cy^2 + \dots.$$

Find A, B, and C, and use them to find the differential equation on the center manifold to third order. Use polar coordinates to see what its flow is. This should enable you to describe the phase portrait near the origin, which is determined by the flow on the center manifold and the fact that there is a stable manifold but no unstable manifold.