MA 732 Homework 4

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1. Let B be an invertible $n \times n$ matrix, and let $G : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 with $\sup_x \|DG(x)\| < \frac{1}{\|B^{-1}\|}$. Define $F : \mathbb{R}^n \to \mathbb{R}^n$ by $F(x) = Bx + G(x)$. Prove: F has a C^1 inverse.

   Method: We want to solve the equation $F(x) = y$ for x in terms of y. Rewrite this equation as $Bx + G(x) = y$, then as $x = B^{-1}(y - G(x))$. Define $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $T(x, y) = B^{-1}(y - G(x))$. We have shown that for a given $y \in \mathbb{R}^n$, $F(x) = y$ if and only if $x = T(x, y)$, i.e., if and only if x is a fixed point of $T(\cdot, y)$. Now you can use the Contraction Mapping Theorem with Parameters to prove the result. You will need to show that for each $y \in \mathbb{R}^n$, $T(\cdot, y)$ is a contraction of $\mathbb{R}^n$.

2. **Implicit Function Theorem**. Let X and Y be Banach spaces, let U be an open subset of X, and let V be an open subset of Y. Let $f : U \times V \to Y$ be C^1. Let $(x_0, y_0) \in U \times V$. Assume that $f(x_0, y_0) = 0$ and that $D_2 f(x_0, y_0)$ is invertible. Show that the exist neighborhoods U_0 of x_0 and V_0 of y_0 such that for each $x \in U_0$ there is a unique $y \in V_0$ such that $f(x, y) = 0$. Moreover, if we write $y = g(x)$, then g is C^1.

   Method: Let $A = D_2 f(x_0, y_0)$. Define $T : U \times V \to Y$ by $T(x, y) = y - A^{-1} f(x, y)$. Notice that T is C^1.

   (a) Show that $T(x, y) = y$ if and only if $f(x, y) = 0$.

   Choose $\delta > 0$ such that if $\|x - x_0\| < \delta$ and $\|y - y_0\| \le \delta$, then $\|D_2 f(x, y) - A\| < \frac{1}{2\|A^{-1}\|}$. Now choose ϵ, $0 < \epsilon \le \delta$, such that if $\|x - x_0\| < \epsilon$, then $\|f(x, y_0)\| \le \frac{\delta}{2\|A^{-1}\|}$. Let $U_0 = \{x : \|x - x_0\| < \epsilon\}$ and let $V_0 = \{y : \|y - y_0\| \le \delta\}$.

   (b) Show that if $x \in U_0$ and $y \in V_0$ then $T(x, y) \in V_0$. Suggestion:

   $$\begin{aligned}\|T(x,y) - y_0\| &= \|y - A^{-1} f(x,y) - y_0\| \\ &= \|A^{-1}\big(Ay - f(x,y) - Ay_0 + f(x,y_0)\big) - A^{-1} f(x,y_0)\|.\end{aligned}$$

   (c) Show that if $x \in U_0$ and y, $y' \in V_0$, then $\|T(x,y) - T(x,y')\| \le \frac{1}{2}\|y - y'\|$. Suggestion:

   $$\begin{aligned}\|T(x,y) - T(x,y')\| &= \|y - A^{-1} f(x,y) - y' + A^{-1} f(x,y')\| \\ &\le \|A^{-1}\| \|Ay - f(x,y) - Ay' + f(x,y')\|.\end{aligned}$$

   (d) Explain how the theorem now follows from the Contraction Mapping Theorem with Parameters.

(e) Now that you know that g is differentiable, use the formula $f(x, g(x)) = 0$ to derive a formula for $Dg(x)$.