MA 732 Homework 3

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1. Let $Y_1, \ldots, Y_k$ by Banach spaces.

   (a) Show that $Y_1 \times \ldots \times Y_k$, with the norm $\|(y_1, \ldots, y_k)\| = \max_{1 \le i \le k} \|y_i\|_i$, is a Banach space. (In other words, show that the "norm" just defined is really a norm, and that $Y_1 \times \ldots \times Y_k$ with this norm is complete.)

   (b) Define $\Pi_i : Y_1 \times \ldots \times Y_k \to Y_i$ by $\Pi_i(y_1, \ldots, y_k) = y_i$. Show that Π_i is a bounded linear map with norm 1.

   (c) Let X be another Banach space, and let $f : X \to Y_1 \times \ldots \times Y_k$ be a map, $f(x) = (f_1(x), \ldots, f_k(x))$. Show that f is continuous if and only if each f_i is continuous. (It may simplify things to note that $f_i = \Pi_i \circ f$.)

   (d) Again let X be another Banach space, and let $f : X \to Y_1 \times \ldots \times Y_k$ be a map, $f(x) = (f_1(x), \ldots, f_k(x))$. Assume that each f_i is differentiable at a point $x \in X$. Show that f is differentiable at x, and

$$Df(x)h = (Df_1(x)h, \ldots, Df_k(x)h).$$

2. Suppose $f : X \to Y$ satisfies (1) f is differentiable at x, and (2) for all x' in some neighborhood of x, $\|f(x') - f(x)\| \le \lambda \|x' - x\|$. Show that $\|Df(x)\| \le \lambda$.

3. Let $A(t)$ be a 3×3 matrix depending differentiably on t, and let $f(t)$ and $g(t)$ be 3×1 vectors depending differentiably on t. Compute $\frac{d}{dt}\left(A(t)\left(f(t) \times g(t)\right)\right)$ using the generalized product rule. (Here $\times$ is cross product.)

4. Consider the map $F : C^0([a,b], \mathbb{R}^n) \to \mathbb{R}$, $F(\phi) = \int_a^b \phi(t) \cdot \phi(t)\, dt$. Write F as a composition of *three* maps, each of which is linear or bilinear. Then use the chain rule and generalized product rule to compute $DF(\phi)$.