MA 732 Homework 1

S. Schecter

January 14, 2009, corrected January 21, 2009

1. In class we studied the differential equation

$$\dot{x} = (a \cos t + b)x - x^3, \quad a > 0,\, b > 0. \tag{1}$$

Let $\phi(t, y)$ denote the solution with $\phi(0, y) = y$. We proved that there is a number $y_0 > 0$ such that $\phi(2\pi, y_0) = y_0$. Since $x \equiv 0$ is also a solution and solutions can't cross, $\phi(t, y_0) > 0$ for all t. In this problem we will show that $\phi(t, y_0)$ is an attracting periodic solution by showing that $\frac{\partial \phi}{\partial y}(2\pi, y_0) < 1$.

To simplify the notation, let $x(t) = \phi(t, y_0)$ and $z(t) = \frac{\partial \phi}{\partial y}(t, y_0)$. Then $z(t)$ satisfies the linear differential equation

$$\dot{z} = \left(a \cos t + b - 3x(t)^2\right) z, \quad z(0) = 1. \tag{2}$$

Therefore

$$z(t) = \exp\left(\int_0^t a \cos s + b - 3x(s)^2 \, ds\right). \tag{3}$$

(a) Use formula (3) to show that $z(2\pi) = e^{-4b\pi} < 1$.
Hint: From (1),

$$\frac{\dot{x}(t)}{x(t)} = a \cos t + b - x(t)^2. \tag{4}$$

Use (4) to substitute for $x(s)^2$ in (3).

(b) Could there be *two* values of $y_0 > 0$ such that $\phi(2\pi, y_0) = y_0$? Explain using the graph of the Poincaré map.

2. Show that the differential equation $\dot{x} = -x^5 + c(t)$, where $c(t)$ is a 2π-periodic continuous function, has a 2π-periodic solution. Show that any such solution is asymptotically stable. Use the graph of the Poincaré map to explain why this implies that there is only one 2π-periodic solution.

3. Suppose that $a(t)$ is 2π-periodic with $0 < a(t) < 1$ for all t. Show that the differential equation $\dot{x} = x(x - a(t))(1 - x)$ has at least three 2π-periodic solutions. Hint: Show that $x(t) \equiv 0$ and $x(t) \equiv 1$ are asymptotically stable 2π-periodic solutions, and use the graph of the Poincaré map to explain why this implies that there is a 2π-periodic solution between them.

4. *Variation of constants formula for nonautonomous linear equations.* Consider $\dot{x} = A(t)x$ with $x \in \mathbb{R}^n$ and $A(t)$ an $n \times n$ matrix that depends continuously on t. Let $\Phi(t)$ be a fundamental matrix solution. Let $h : \mathbb{R} \to \mathbb{R}^n$ be continuous. Show that the solution of $\dot{x} = A(t)x + h(t)$, $x(0) = x_0$, is

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)h(s)\,ds.$$

Suggestion: just check that it works!

5. *Fredholm alternative.* Suppose that $a(t)$ and $b(t)$ are 2π-periodic continuous functions, and let $a_0 = \int_0^{2\pi} a(s)\ ds$. Show the following properties of the differential equation $\dot{x} = a(t)x + b(t)$.

   (a) If $a_0 \neq 0$, then there is a unique 2π-periodic orbit. It is asymptotically stable if $a_0 < 0$, and asymptotically unstable if $a_0 > 0$.

   (b) Suppose $a_0 = 0$. Let $c_0 = \int_0^{2\pi} \exp\{\int_s^{2\pi} a(u)\ du\}\, b(s)\, ds$.

      i. If $c_0 = 0$, then every solution is 2π-periodic.

      ii. If $c_0 \neq 0$, then every solution is unbounded.

   Hint: Show using the variation of constants formula that the Poincaré map is

$$P(\xi) = e^{a_0}\xi + \int_0^{2\pi} \exp\{\int_s^{2\pi} a(u)\,du\}\, b(s)\,ds,$$

   and $P(\xi) = \xi$ if and only if $(1 - e^{a_0})\xi = c_0$.

6. *Riccati equation.* Suppose that $a(t)$ and $b(t)$ are 2π-periodic continuous functions. Prove that the Riccati equation

$$\dot{x} = b(t) + a(t)x - x^2$$

has at most two 2π-periodic solutions. Hint: Suppose that $\phi(t)$ is a 2π-periodic solution. If $x(t)$ is another solution, let $y(t) = x(t) - \phi(t)$. Show that

$$\dot{y} = c(t)y - y^2,$$

where $c(t) = a(t) - 2\phi(t)$. Then let $w(t) = \frac{1}{y(t)}$. Show that

$$\dot{w} = -c(t)w + 1$$

Use the Fredholm Alternative to discuss separately the cases $\int_0^{2\pi} c(t)\,dt \neq 0$ and $\int_0^{2\pi} c(t)\ dt = 0$.