TRAPPING REGION FOR LOTKA-VOLTERRA WITH A LOGISTIC MODIFICATION

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Yesterday in class we showed the existence of a triangular trapping region for this equation. However, the argument was incomplete because I wrongly *assumed* that a certain parabola had to open upward. Here is a better argument, in which the fact that the parabola opens upward is a *consequence* of other assumptions.

System (sec. 4.3.2 in the text with different letters for the constants):

$$\begin{aligned} \dot{x} &= x(1 - y - \eta x), \\ \dot{y} &= \gamma y(x - 1), \end{aligned}$$

with η and γ positive. We assumed $\eta < 1$ in order to have an interior equilibrium, but this is not needed to get a trapping region.

The desired trapping region is the triangle bounded on the left by $x = 0$, below by $y = 0$ (both these lines are invariant), and above by the line

$$H(x, y) = \frac{x}{A} + \frac{y}{B} = 1, \quad A,\ B > 0.$$

For a trapping region we need $\nabla H \cdot (\dot{x}, \dot{y}) \leq 0$ when $H = 1$ and $0 \leq x \leq A$. We have

$$\nabla H \cdot (\dot{x}, \dot{y}) = \frac{1}{A} x(1 - y - \eta x) + \frac{1}{B} \gamma y(x - 1).$$

When $H = 1$, $y = B - \frac{B}{A}x$. Substituting this expression into the above formula and simplifying, we find that when $H = 1$, $\nabla H \cdot (\dot{x}, \dot{y})$ is given by

$$h(x) = \frac{1}{A}\left(\left(\frac{B}{A} - \eta - \gamma\right)x^2 + \Big(1 - B + \gamma A + \gamma\Big)x - A\gamma\right).$$

The graph of $z = h(x)$ is a parabola.

For a trapping region we need

$$h(x) \leq 0 \text{ for } 0 \leq x \leq A. \tag{0.1}$$

Notice that

(1) $h(0) = -\gamma < 0$, i.e., the z-intercept of the parabola $z = h(x)$, is negative.

We claim that if A and B are large enough that

$$A \geq \frac{1}{\eta} \text{ and } B > (\eta + \gamma)A, \tag{0.2}$$

then (0.1) is satisfied.

Since $B > (\eta + \gamma)A$,

(2) the coefficient of x^2 in $h(x)$ is positive, i.e., the parabola $z = h(x)$ opens upward.

In addition,

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$$(3)\ \ h(A) = \tfrac{1}{A}\Bigg(\Big(\tfrac{B}{A} - \eta - \gamma\Big)A^2 + \Big(1 - B + \gamma A + \gamma\Big)A - \gamma A\Bigg) = -\eta A + 1 \leq 0.$$

The inequality follows from the assumption $A \geq \frac{1}{\eta}$.

Now (1), (2), and (3) imply (0.1).