MA 532 Problems for Chapter 5

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1. Consider the system in problem 9 of chapter 4, which you have already worked with. Assume that $K_2K_3 - K_1K_4 > 0$, so there is an equilibrium (x_*, y_*) with $x_* > 0$ and $y_* > 0$. It is the solution of the pair of equations

$$1 - \frac{x}{K_1} + \frac{y}{K_3} = 0, \quad 1 + \frac{x}{K_2} - \frac{y}{K_4} = 0. \tag{1}$$

In this problem I suggest that you not find a formula for (x_*, y_*) and that you never put two fractions over a common denominator.

   (a) Find the 2×2 matrix $Df(x, y)$.

   (b) Substitute $(x, y) = (x_*, y_*)$ into your answer to part (a) to obtain $Df(x_*, y_*)$, and simplify **using the fact that** (x_*, y_*) **is the solution of** (1).

   (c) Show that trace$Df(x_*, y_*) < 0$ and det$Df(x_*, y_*) > 0$. Consult Proposition 2.4.4 in the text or your notes to see that this implies that (x_*, y_*) is an attractor.

2. Using the previous problem as a guide, do problem 2(b') in chapter 5.

3. Problem 2(d) in chapter 4. (The Lorenz equations are in problem 8(e) of Chapter 4.) Use linearization. Also: for $\rho > 1$ show that the equilibrium at the origin is hyperbolic, and give the dimensions of the stable and unstable manifolds.

4. (Based on problems 3(a) and (b) in chapter 5.) Consider the differential equation $\ddot{x} + b\dot{x} - x + x^3 = 0$. Written as a system, it is

$$\dot{x} = y, \quad \dot{y} = x - x^3 - by. \tag{2}$$

   (a) For $b = 0$, find a function of the form $H(x, y) = \frac{1}{2}y^2 + V(x)$ that is constant on solutions. Choose V so that $V(0) = 0$.

   (b) For $b = 0$, use your answer to (a) to draw the phase portrait.

   (c) For $b > 0$, show that $\dot{H} \leq 0$.

   (d) Let

$$U_1 = \{(x, y) : -\sqrt{2} < x < 0 \text{ and } H(x, y) < 0\},$$
$$U_2 = \{(x, y) : 0 < x < \sqrt{2} \text{ and } H(x, y) < 0\}.$$

   For $b > 0$, use LaSalle's invariance principle to show that every solution that starts in U_1 approaches the equilibrium $(-1, 0)$, and every solution that starts in U_2 approaches the equilibrium $(1, 0)$.

(e) For $b > 0$, use LaSalle's invariance principle to show that every solution approaches one of the three equilibria.

5. Show that for $\rho < 1$, the function $L(x, y, z) = \frac{x^2}{\sigma} + y^2 + z^2$ is a strict Liapunov function for the equilibrium at the origin of the Lorenz equations. (You will need to know how to show that a quadratic form $ax^2 + bxy + cy^2$ is positive definite, i.e., positive except at $(x, y) = (0, 0)$. If you can't figure this out, please ask.) Then use LaSalle's invariance principle to show that the origin is globally asymptotically stable.

6. Problem 4(a) in chapter 5. Use problem 1 of this homework set together with the slope of the unstable manifold of the origin.

7. Problem 4(b) in chapter 5. The system is again the one in problem 9 of chapter 4.

8. There is an alternate approach to Theorem 5.1.1 that uses a fixed point argument instead of Grönwall's inequality. We will show only stability of the equilibrium; some more work is needed to show asymptotic stability.

   As in Theorem 5.1.1, we consider $\dot{x} = Ax + r(x)$, $x \in \mathbb{R}^n$, where all eigenvalues of A have real part less than $-a < 0$. We assume r is C^1, $r(0) = 0$, and $Dr(0) = 0$. We want to show that the equilibrium at 0 is stable. In other words, given $\delta > 0$, we want to find a number $\gamma > 0$ such that if $|b| \leq \gamma$, then the solution $x(t)$ with $x(0) = b$ satisfies $|x(t)| \leq \delta$ for $0 \leq t < \infty$.

   Any solution of $\dot{x} = Ax + r(x)$, $x(0) = b$, satisfies

$$x(t) = e^{At}b + \int_0^t e^{A(t-s)} r(x(s))\, ds. \tag{3}$$

   Let $C^0([0, \infty), \mathbb{R}^n)$ denote the space of *bounded* continuous functions from $[0, \infty)$ to $\mathbb{R}^n$ with the sup norm:

$$\|x\| = \sup_{0 \leq t < \infty} |x(t)|.$$

   This is a Banach space. Define a mapping $T : C^0([0, \infty), \mathbb{R}^n) \to C^0([0, \infty), \mathbb{R}^n)$ by $Tx =$ the right hand side of (3). (You should check that if $x(t)$ is bounded on $0 \leq t < \infty$, then $Tx(t)$ is bounded on $0 \leq t < \infty$.) The solution we are looking for is a fixed point of T.

   Let

$$B = \{x \in C^0([0, \infty), \mathbb{R}^n) : x(0) = b \text{ and } \|x\| \leq \delta\},$$

   a closed set. (Of course the definition of B depends on δ and b.) Given $\delta > 0$, we wish to find $\gamma > 0$ such that if $|b| \leq \gamma$, then T maps B into itself and is a contraction. If we can do this, the fixed point of the mapping is what we want.

   There is a number $K > 0$ such that $\|e^{At}\| \leq Ke^{-at}$ for $t \geq 0$.

   Here is a useful fact. Since r is C^1 and $Dr(0) = 0$, for each $\epsilon > 0$ there exists $\eta > 0$ such that if $|x| \leq \eta$, then $\|Dr(x)\| \leq \epsilon$. Then we have two nice estimates:

   - For $|x| \leq \eta$, by the Mean Value Theorem, $r(x) - r(0) = \left(\int_0^1 Dr(tx)\, dt\right) x$. Since $r(0) = 0$, $|r(x)| \leq \epsilon |x|$.

- For $\|x_1\| \leq \eta$ and $\|x_2\| \leq \eta$, by the Mean Value Theorem, $r(x_2) - r(x_1) = \left(\int_0^1 Dr(x_1 + t(x_2 - x_1))\, dt\right)(x_2 - x_1)$, so $|r(x_2) - r(x_1)| \leq \epsilon |x_2 - x_1|$

It is enough to prove the result for small δ. We will only use δ small enough so that $|x| \leq \delta$ implies $\|Dr(x)\| \leq \frac{a}{2K}$.

(a) Show that for such a δ, if $|b| \leq \frac{\delta}{2K}$, then T maps B into B.

(b) What additional condition must be place on δ so that T is a contraction?