MA 532 Homework 7

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1. Sec. 4.6 problem 1(l). It's really *Theorem* 4.4.4 that you are asked to prove. Suggestion:

$$x(t) - y(t) = \int_0^t F(x(s)) - G(y(s))\, ds = \int_0^t F(x(s)) - F(y(s)) + F(y(s)) - G(y(s))\, ds.$$

   You will need the generalized Grönwall inequality.

2. Consider the system

$$\begin{aligned} \dot{r} &= f_1(r, \theta) = r(1-r), \\ \dot{\theta} &= f_2(r, \theta) = 1, \end{aligned}$$

   with initial conditions $(r(0), \theta(0)) = (b, c)$. Let ϕ be the flow:

$$\phi\big(t, (b,c)\big) = \big(r(t,(b,c)), \theta(t,(b,c))\big)$$

   (a) Calculate ϕ. Answer:

$$\phi\big(t, (b,c)\big) = \big(r(t,(b,c)), \theta(t,(b,c))\big) = \left(\frac{b}{b + (1-b)e^{-t}},\ c + t\right).$$

   (b) Use the above formula to calculate

$$D_{(b,c)}\phi\big(t,(b,c)\big) = \begin{pmatrix} \frac{\partial r}{\partial b} & \frac{\partial r}{\partial c} \\ \frac{\partial \theta}{\partial b} & \frac{\partial \theta}{\partial c} \end{pmatrix}$$

   (c) We know that $D_{(b,c)}\phi\big(t,(b,c)\big)$ is the solution of the linear differential equation

$$\dot{\Phi} = \begin{pmatrix} \frac{\partial f_1}{\partial r}\Big(\phi\big(t,(b,c)\big)\Big) & \frac{\partial f_1}{\partial \theta}\Big(\phi\big(t,(b,c)\big)\Big) \\ \frac{\partial f_2}{\partial r}\Big(\phi\big(t,(b,c)\big)\Big) & \frac{\partial f_2}{\partial \theta}\Big(\phi\big(t,(b,c)\big)\Big) \end{pmatrix} \Phi, \quad \Phi(0) = I.$$

   Use this formula to calculate $D_{(b,c)}\phi\big(t,(1,c)\big)$ and compare to the previous formula.

3. In the previous problem, note that, because $\frac{\partial f_1}{\partial \theta}\Big(\phi\big(t,(b,c)\big)\Big) = 0$, $\frac{\partial r}{\partial b}\Big(\phi\big(t,(b,c)\big)\Big)$ satisfies the linear differential equation

$$\dot{x} = \frac{\partial f_1}{\partial r}\Big(\phi\big(t,(b,c)\big)\Big)x, \quad x(0) = 1.$$

With this in mind, do problem 13 in section 4.6.2.

4. Suppose $H : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is C^1 and has $c \in \mathbb{R}^{n-k}$ as a regular value. We showed in lecture that $S = H^{-1}(c)$ is a C^1 k-dimensional submanifold of $\mathbb{R}^n$. Let $p \in S$. Prove: $T_pS = \operatorname{Ker}\big(DH(p)\big)$.

   Suggested outline:

   (a) $\operatorname{Ker}\big(DH(p)\big)$ has dimension k. (Why?)

   (b) Since S is a k-dimensional manifold, T_pS has dimension k.

   (c) Therefore we need only show that $T_pS \subset \operatorname{Ker}\big(DH(p)\big)$.

   (d) Let V be an open subset of $\mathbb{R}^k$, and let $G : V \to S$ be a submanifold chart with $G(x_0) = p$. Let $w \in T_pS$. We must show that $DH(p)w = 0$. There is a vector $v \in T_{x_0}\mathbb{R}^k$ such that $DG(x_0)v = w$. Consider the curve $\phi(t) = x_0 + tv$ in $\mathbb{R}^k$ and the corresponding curve $z = G(\phi(t))$ in S.

5. Show that a k-dimensional subspace of $\mathbb{R}^n$ is a k-dimensional submanifold of $\mathbb{R}^n$.

   Suggested outline:

   (a) Let S be a k-dimensional subspace of $\mathbb{R}^n$. Let $v_1, \dots, v_k$ be a basis. Form the matrix

   $$A = \begin{pmatrix} v_1 & \dots & v_k \end{pmatrix},$$

   an $n \times k$ matrix.

   (b) By reordering the coordinates, we can assume that

   $$A = \begin{pmatrix} B \\ C \end{pmatrix},$$

   where B is $k \times k$, C is $(n-k) \times k$, and B is invertible. (Why?)

   (c) Let E be an invertible $k \times k$ matrix. Then the columns of AE form a basis for S. (Why?)

   (d) In particular, let $E = B^{-1}$. The rest of the proof is left to you. You should show that S is the graph of a function $g : \mathbb{R}^k \to \mathbb{R}^{n-k}$. (Only one chart is needed.)

6. Let $S^1 = \{(x,y) : x^2 + y^2 = 1\}$. There is a well-known function $\alpha : \mathbb{R} \to S^1$ defined by $\alpha(\theta) = (\cos\theta, \sin\theta)$. Suppose $\beta(\theta)$ is a C^1 2π-periodic function from $\mathbb{R}$ to $\mathbb{R}$. Show that β can be used to define a C^1 function $\gamma : S^1 \to \mathbb{R}$ such that $\gamma(\alpha(\theta)) = \beta(\theta)$. (You need to show: (a) γ is well-defined, and (2) γ is C^1. To show the latter, recall the system of four charts on S^1 that we used in class. You should show that γ is C^1 in each chart. Instead just show this for the chart $H_+ : (-1,1) \to S^1$ defined by $H_+(y) = (\sqrt{1-y^2}, y)$, and say that the other charts are similar.)

7. Let $S = \{(x, y, z) : (x-2)^2 + z^2 = 1 \text{ and } y = 0\}$, a circle in the plane $y = 0$. Let T denote the surface of revolution obtained by rotating S about the z-axis. (T is a torus.) Show that T is a 2-dimensional submanifold of $\mathbb{R}^3$.

8. Let $D = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$. Consider the following differential equation on $\mathbb{R}^n \times \mathbb{R}^n$:

$$\dot{x} = f(x) + h(x-y), \quad \dot{y} = f(y) + g(x-y),$$

where f, g, h are C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $g(0) = h(0)$. Prove that D is invariant. Suggestion: Let $H(x, y) = x - y$. Then $D = H^{-1}(0)$.

9. Section 4.6 problem 8e. Assume σ, ρ, and β are positive.

Suggestions:

Let $H(x, y, z) = x^2 + y^2 + (z - \rho - \sigma)^2$. You must show that for large A, $H = A^2$ implies $\nabla H \cdot F \leq 0$. ($F(x, y, z)$ = right hand side of the system.)

(a) Show that $\nabla H \cdot F = -2\sigma x^2 - 2y^2 - 2\beta(z - \rho - \sigma)^2 - 2\beta(\rho + \sigma)(z - \rho - \sigma)$.

(b) Using (a), show that $H = A^2$ implies

$$\nabla H \cdot F \leq -2\min(\sigma, 1, \beta)A^2 - 2\beta(\rho + \sigma)(z - \rho - \sigma).$$

(c) Complete the argument.

10. Consider the system in section 4.6, problem 9. The r_i and K_j are all positive.

(a) There are two cases: either the nullclines intersect in the interior of the first quadrant, or they do not. For each case, draw the nullclines, draw typical vectors on the nullclines, and draw typical vectors in the open regions between the nullclines, as in Figure 4.4(a). As in that figure, limit your attention to the first quadrant.

(b) Give conditions on $K_1, \ldots, K_4$ that distinguish between the two cases.

(c) For one of the two cases it is possible to draw large rectangular trapping regions in the first quadrant. Which case is it? For that case, add a large rectangular trapping region to your sketch.

Remark: for the case in which large rectangular trapping regions exist, every point in the first quadrant is in such a compact trapping region, so for all $x_0 \geq 0$ and $y_0 \geq 0$, $\phi(t, (x_0, y_0))$ is defined for all $t \geq 0$.