MA 532 Homework 6

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October 10, 2012, corrected October 16, 2012

1. Consider the linear differential equation $\dot{x} = Ax + f(t)$ with $x \in \mathbb{R}^n$ and f continuous. Suppose there exist numbers $A > 0$, $\mu > 0$, and $\nu > 0$ such that (1) all eigenvalues of A have real part less than or equal to $-\mu$, and (2) for $t \geq 0$, $|f(t)| \leq Ae^{-\nu t}$. Show that all solutions are bounded on $0 \leq t < \infty$. (Proposition 2.4.3 and formula (2.39) should help.)

2. Section 3.5 problem 1(c), with the following suggestions and changes:

   (a) $F(x) = \sqrt{|x|}$: Is $|F(x) - F(0)|/|x - 0|$ bounded for x near 0?

   (b) Of course $F(x) = |x|$ is not differentiable at 0, you can ignore that part of the question. You need to find a bound for $|F(x) - F(y)|/|x - y|$ that works for all $x \neq y$.

   (c) $F(x) = x^2$ is locally Lipschitz because it is C^1, you can ignore that part of the question. Show that $|F(x) - F(y)|/|x - y|$ can be arbitrarily large.

3. Section 3.5 problem 1(h). You must show that if $|x - x_k| < \frac{\delta_k}{2}$ and $|y - x_k| \geq \delta_k$, then $|y - x| > \frac{\delta_k}{2}$, without relying on a picture. Here x, x_k, and y are points in $\mathbb{R}^n$.

4. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^l$ be Lipschitz. Prove: $G \circ F$ is Lipschitz. Explain how to do Section 3.5 problem 2(b) using this result.

5. Section 3.5 problem 5(b).

6. (Based on Section 3.5 problem 6(b).) $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is *locally Lipschitz in x* if for each $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, there exist positive numbers L, δ, and η such that:

   If $|x_1 - x_0| < \delta$, $|x_2 - x_0| < \delta$, and $|t - t_0| < \eta$, the $|F(x_1, t) - F(x_2, t)| \leq L|x_1 - x_2|$.

   Notice that the points (x_1, t) and (x_2, t) differ only in their x-coordinate.

Consider $\dot{x} = F(x,t)$ with $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *locally Lipschitz in* x and *continuous*. Prove: The initial value problem $\dot{x} = F(x,t)$, $x(t_0) = x_0$, has a solution defined on an interval of the form $(t_0 - \eta, t_0 + \eta)$. You should be able to do this by mimicking the proof of the Existence Theorem for autonomous equations.

7. Section 4.6 problem 3.