MA 532 Homework 5

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1. Prove: $\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$. Hint: $A(t)A^{-1}(t) = I$ for all t.

2. Consider the linear differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 1 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Find the state transition matrix $\Psi(t,s)$. Recall that $\Psi(t,s)$ is a matrix whose entries depend on t and s, with the property that for any solution and any times t and s, $x(t) = \Psi(t,s)x(s)$. Suggestion: If you can find a fundamental matrix solutions (the columns are two linearly independent solutions) $\Phi(t)$, then $\Psi(t,s) = \Phi(t)\Phi^{-1}(s)$. You can find the general solution by first solving the $\dot{x}_1$ equation (since it only depends on x_1), then substituting into the $\dot{x}_2$ equation.

3. Let A be a 4×4 real matrix, and consider the linear differential equation $\dot{x} = Ax$. Suppose that A has the complex eigenvalues $a \pm bi$ repeated twice, and the complex Jordan form is

$$\begin{pmatrix} a+bi & 1 & 0 & 0 \\ 0 & a+bi & 0 & 0 \\ 0 & 0 & a-bi & 1 \\ 0 & 0 & 0 & a-bi \end{pmatrix}.$$

Let $v + iw$ be an eigenvector for the eigenvalue $a + bi$, and let $y + iz$ satisfy $(A - (a+bi)I)(y+iz) = v + iw$. Show that

$$A\begin{pmatrix} v & w & y & z \end{pmatrix} = \begin{pmatrix} v & w & y & z \end{pmatrix} \begin{pmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}.$$

(Thus the matrix $\begin{pmatrix} v & w & y & z \end{pmatrix}$ can be used to conjugate $\dot{x} = Ax$ to real Jordan form.) Suggestion: To get started, you know that $A(v+iw) = (a+bi)(v+iw)$. Multiply out and equate real and imaginary parts to find Av and Aw. Actually, we did this in class.

4. For each of the following linear differential equations, use eigenvalues and eigenvectors to sketch the phase portrait. If typical solutions approach the origin as $t \to \infty$ or as $t \to -\infty$, be sure that your picture clearly shows the line through the origin along which they approach. Exception: if the eigenvalues are complex, you don't need to draw a picture. Just state whether solutions rotate clockwise or counterclockwise (with justification), and state whether they spiral in, spiral out, or close up.

   (a) $\dot{x} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} x$

   (b) $\dot{x} = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} x$

   (c) $\dot{x} = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} x$

   (d) $\dot{x} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} x$

5. Consider the initial value problem

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}.$$

   (a) Find the solution. (Suggestion: solve the $\dot{x}_2$ equation first.)

   (b) Show that if $x_2^0 \neq 0$,

$$\frac{x_2(t)}{x_1(t)} \to 0 \textit{ from the right} \text{ as } t \to +\infty.$$

   (c) Explain the following: if $\lambda < 0$, then for $x_2^0 > 0$, the solution approaches the origin from the right and becomes close to the x_1-axis ; for $x_2^0 < 0$, the solution approaches the origin from the left and becomes close to the x_1-axis.