MA 532 Homework 4

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1. If A is a square matrix, we know that there is a matrix in Jordan form J and a change-of-coordinates matrix P such that $P^{-1}AP = J$. We also know that $e^{tA} = Pe^{tJ}P^{-1}$.

   For each of the following matrices A, find J, e^{tJ}, and a change-of-coordinates matrix P. From this information you could compute e^{tA}, but don't do it.

   (a) $\begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$

   Comment: There is a triple eigenvalue 1 and just one eigenvector v_1 (up to multiplication by a number). You need to find v_1, v_2 such that $(A - I)v_2 = v_1$, and v_3 such that $(A - I)v_3 = v_2$. Unless you are very unlucky, you should be able to pick v_3 at random at random and then use it to find v_2 and v_1.

   (d) $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & -2 \\ -1 & 0 & 2 \end{pmatrix}$

   (e) $\begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

2. Chapter 2 problem 6. First find A. Then show (1) for fixed $\delta > 0$, as $\epsilon \to 0$, $\|A\| \to \infty$, and (2) for fixed $\epsilon > 0$, as $\delta \to 0$, $\|A\| \to \infty$. To do this, you do not need to calculate $\|A\|$. If you take a particular v and calculate $\|Av\|$ and $\|v\|$, then you know that $\|A\| \geq \|Av\|/\|v\|$.

3. Chapter 2 problem 8. In this problem, $< x, y >= x \cdot y$ (dot product) and $|x|$ is the Euclidean norm, so $|x|^2 =< x, x >= x \cdot x$. Follow the hint until you get

$$\frac{d}{dt}u(t) \leq -2\gamma u(t).$$

After that, an alternative approach to the one in the hint is to rewrite the inequality as

$$\frac{\dot{u}}{u} \leq -2\gamma$$

and integrate both sides from 0 to t.

4. Let A be an invertible $n \times n$ matrix, and let b be a vector in $\mathbb{R}^n$. Consider the differential equation

$$\dot{x} = Ax + b, \quad x(0) = x_0. \tag{1}$$

   (a) Show that the change of coordinates $u = x + A^{-1}b$ converts this differential equation to

$$\dot{u} = Au, \quad u(0) = x_0 + A^{-1}b. \tag{2}$$

   (b) The solution of (2) is of course $u = e^{tA}u(0)$. By undoing the change of coordinates find the solution of (1).

5. Let A be an $n \times n$ matrix, and let f be a continuous function from an interval J that contains 0 to $\mathbb{R}^n$. Consider the differential equation

$$\dot{x} = Ax + f(t), \quad x(0) = x_0. \tag{3}$$

Our formula for the solution of an inhomogeneous linear differential equation (the "variation of constants" or "variation of parameters" formula), in the special case where A is constant, says that the solution is

$$x = e^{tA}x_0 + \int_0^t e^{(t-s)A} f(s)\, ds. \tag{4}$$

Suppose A is invertible and $f(t)$ is a constant vector b. Try to rewrite (4) to equal your solution to 4(b).