MA 532 Homework 2

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1. Sketch the phase portrait.

   (a) $\dot{x} = x - x^3$

   (b) $\dot{x} = 1 + x^2$

   (c) $\dot{x} = x^2 - x^3$

   (d) $\dot{x} = 1 - \cos x$

2. The population of fish in a lake satisfies the differential equation

   $$\dot{x} = ax - bx^2,$$

   where x is the number of fish, t is time in years, $a > 0$ and $b > 0$ are constants. The manager of the lake proposes to allow fishing at a rate of h fish per year. Complete the following sentence: If h is greater than [fill in the blank with a number that depends on a and b], the fish population will crash. Justify your answer using phase portraits.

   Suggestion: The new differential equation is

   $$\dot{x} = ax - bx^2 - h.$$

   The phase portrait depend on the number of roots of the equation $ax - bx^2 - h = 0$.

3. Sketch the phase portrait. Be sure to include equilibria.

   (a) $\ddot{x} + x - x^2 = 0$

   (b) $\ddot{x} - x + x^3 = 0$

4. Consider the linear differential equation

   $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are continuous on an interval J. Suppose we know one solution

$$\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} \tag{2}$$

with $\phi_1(t_0) \neq 0$ and $\phi_2(t_0) \neq 0$. Let $\alpha(t)$ and $\beta(t)$ be differentiable functions defined on J such that

$$\begin{pmatrix} \alpha(t)\phi_1(t) \\ \beta(t)\phi_2(t) \end{pmatrix} \tag{3}$$

is a second, linearly independent solution on J.

(a) Explain why for all $t \in J$, $\alpha(t) \neq \beta(t)$.

(b) Let

$$W(t) = \det \begin{pmatrix} \alpha(t)\phi_1(t) & \phi_1(t) \\ \beta(t)\phi_2(t) & \phi_2(t) \end{pmatrix} = (\alpha(t) - \beta(t))\phi_1(t)\phi_2(t). \tag{4}$$

Show that

$$\begin{pmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \end{pmatrix} = W(t) \begin{pmatrix} -b(t)/\phi_1(t)^2 \\ c(t)/\phi_2(t)^2 \end{pmatrix}. \tag{5}$$

(c) Show that

$$W(t) = (\alpha(t_0) - \beta(t_0))\phi_1(t_0)\phi_2(t_0)e^{\int_{t_0}^t a(s)+d(s)\,ds}. \tag{6}$$

Hint: Liouville's Formula.

(d) Of course $\alpha(t)$ and $\beta(t)$ are not unique: they can, for example, be multiplied by constants. Thus $W(t_0)$ is arbitrary (but not 0). If you choose, for example, $W(t_0) = 1$, so that

$$W(t) = e^{\int_{t_0}^t a(s)+d(s)\,ds}, \tag{7}$$

and substitute (7) into (5), then the differential equation (5) can be integrated to produce $\alpha(t)$ and $\beta(t)$. There will be two arbitrary constants c_1 and c_2. The fact that $W(t_0) = 1$ will let you solve for one of these constants in terms of the other. The remaining constant can be chosen to make the solution as simple as possible. Use this approach to find a second, linearly independent solution of

$$\dot{x} = \begin{pmatrix} 1 & -1/t \\ 1+t & -1 \end{pmatrix} x, \quad t > 0,$$

given that one solution is $x(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$.