DIFFERENTIABILITY OF THE FLOW

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Theorem 0.1. *Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}^n$ be C^1, and consider the differential equation $\dot{x} = f(x)$, $x \in U$. Let $\phi(t, y)$ be the flow. Then ϕ is C^1 and $D_y\phi(t, y)$ is the solution of the linear differential equation $\dot{\Phi} = Df(\phi(t, y))\Phi$, $\Phi(0) = I$.*

Remarks:

(1) For each (t, y), $D_y\phi(t, y)$ is an $n \times n$ matrix.

(2) With y fixed, $\dot{\Phi} = Df(\phi(t, y))\Phi$ is a nonautonomous linear differential equation. We are looking for the fundamental matrix solution that is principal at $t = 0$.

(3) We have already shown that ϕ is continuous, and that for each $y_0 \in U$ and time T such that $\phi(y_0, t)$ is defined for $0 \le t \le T$, there are numbers $\delta > 0$ and $L > 0$ such that if $|y_1 - y_0| < \delta$, then $|\phi(y_1, t) - \phi(y_0, t)| \le L|y_1 - y_0|$ for $0 \le t \le T$.

Lemma 0.2. *Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}^n$ be C^1, and define $R : U \times U \to \mathbb{R}^n$ by*

$$R(x, z) = \begin{cases} \frac{1}{|z-x|}\Big(f(z) - f(x) - Df(x)(z - x)\Big) & \text{if } z \ne x, \\ 0 & \text{if } z = x. \end{cases}$$

Let K be a compact subset of U, and let $\tilde{K}$ be a compact η-neighborhood of K. Then: given $\epsilon > 0$ there exists $\delta > 0$ such that if x, $z \in \tilde{K}$ and $|z - x| < \delta$, then $|R(x, z)| < \epsilon$.

Proof. For z close to x, the Mean Value Theorem yields:

$$R(x, z) = \frac{1}{|z - x|}\left(\left(\int_0^1 Df\big(x + t(z - x)\big)\, dt\right)(z - x) - Df(x)(z - x)\right)$$
$$= \frac{1}{|z - x|}\left(\int_0^1 \Big(\big(Df\big(x + t(z - x)\big) - Df(x)\Big)dt\right)(z - x).$$

Now $Df(x)$ is a continuous function of x, and on $\tilde{K}$ it is uniformly continuous. Therefore given $\epsilon > 0$ there exists $\delta > 0$ such that if x, $z \in \tilde{K}$ and $|z - x| < \delta$ then $\|Df(z) - Df(x)\| < \epsilon$. For such x, z, with $x \ne z$,

$$|R(x, z)| < \frac{1}{|z - x|}\epsilon|z - x| = \epsilon.$$

□

Proof of Theorem: Fix $y \in U$ and a time T such that $\phi(y, t)$ is defined for $0 \le t \le T$. For $0 \le t \le T$, define

$$g(t, h) = |\phi(t, y + h) - \phi(t, y) - \Phi(t, y)h|,$$

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where $\Phi(t,y)$ is the solution of $\dot\Phi = Df(\phi(t,y))\Phi$, $\Phi(0)=I$. We need to show that $\frac{g(t,h)}{|h|} \to 0$ as $h \to 0$.

We calculate:

$$\begin{aligned} g(t,h) =& \Big| y+h+\int_0^1 f\big(\phi(s,y+h)\big)\,ds - \Big(y+\int_0^1 f\big(\phi(s,y)\big)\,ds\Big) \\ &- \Big(I+\int Df\big(\phi(s,y)\big)\Phi(s,y)\,ds\Big)h \Big| \\ =& \int_0^1 f(\phi(s,y+h)) - f(\phi(s,y)) - Df(\phi(s,y))\Phi(s,y)h\,ds. \end{aligned} \tag{0.1}$$

The definition of R can be rearranged to yield

$$f(z)-f(x) = Df(x)(z-x) + R(x,z)|z-x|.$$

Apply this to (0.1) with $x=\phi(s,y)$ and $z=\phi(s,y+h)$:

$$\begin{aligned} g(t,h) &= \Big| \int_0^t Df(\phi(s,y))\big(\phi(s,y+h)-\phi(s,y)\big) \\ &\quad + R(\phi(s,y),\phi(s,y+h))\cdot|\phi(s,y+h)-\phi(s,y)| - Df(\phi(s,y))\Phi(s,y)h\,ds\Big| \\ &= \Big| \int_0^t Df(\phi(s,y))\big(\phi(s,y+h)-\phi(s,y)-\Phi(s,y)h\big) \\ &\quad + R(\phi(s,y),\phi(s,y+h))\cdot|\phi(s,y+h)-\phi(s,y)|\,ds\Big|. \end{aligned}$$

Let $M \geq \sup\{\|Df(\phi(s,y))\| : 0 \leq s \leq T\}$. The sup is finite since $Df(\phi(s,y))$ is a continuous function of s and the interval $0 \leq s \leq T$ is compact. We shall also take $M>1$.

Then

$$\begin{aligned} g(t,h) &\leq \int_0^t M\big|\phi(s,y+h)-\phi(s,y)-\Phi(s,y)h\big|\,ds \\ &\quad + \int_0^t |R(\phi(s,y),\phi(s,y+h))|\cdot|\phi(s,y+h)-\phi(s,y)|\,ds \\ &= \int_0^t Mg(s,h)\,ds + \int_0^t |R(\phi(s,y),\phi(s,y+h))|\cdot|\phi(s,y+h)-\phi(s,y)|\ ds. \end{aligned}$$

We want to choose $\delta>0$ so that for $|h|<\delta$ and $0 \leq s \leq T$, we have

(1) $|R(\phi(s,y),\phi(s,y+h))| < \frac{\epsilon}{LTe^{MT}}$, and
(2) $|\phi(s,y+h)-\phi(s,y)| \leq L|h|$.

If we can do this, then for $|h|<\delta$ and $0 \leq t \leq T$ we have

$$g(t,h) \leq \int_0^t Mg(s,h)\,ds + \int_0^t \frac{\epsilon}{LTe^{MT}}L|h|\,ds \leq \int_0^t Mg(s,h)\,ds + \frac{\epsilon}{e^{MT}}|h|.$$

Then by Grönwall's inequality,

$$g(t,h) \leq \frac{\epsilon}{e^{MT}}|h|e^{Mt} \leq \epsilon|h| \text{ for } |h|<\delta \text{ and } 0 \leq t \leq T.$$

This proves that $\frac{g(t,h)}{|h|} \to 0$ as $h \to 0$, which proves the Theorem.

Can we find δ so that (1) and (2) are true for $|h| < \delta$ and $0 \leq s \leq T$?

By Lemma 0.2, given $\epsilon > 0$, there exists $\delta_1 > 0$ such that if x, $z \in \tilde{K}$ and $|z - x| < \delta_1$, then $|R(x,z)| < \frac{\epsilon}{LTe^{MT}}$.

By Remark (3), there exists $\delta_2 > 0$ such that if $|h| < \delta_2$, then $|\phi(s, y+h) - \phi(s,y)| \leq L|h|$ for $0 \leq s \leq T$.

Let $\delta = \min\left(\frac{\delta_1}{L}, \delta_2\right)$. If $|h| < \delta$, then both (1) and (2) are true for $0 \leq s \leq T$.