MA 532 Homework 9

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1. For each of the following systems, use the function $L(x, y) = x^2 + y^2$ as a Lyapunov function to show that the origin is an asymptotically stable equilibrium.

   (a) $\dot{x} = y - x^3, \quad \dot{y} = -x - y^3$.

   (b) $\dot{x} = y - xg(x, y), \quad \dot{y} = -x - yg(x, y)$, where g is a C^1 function that is positive on a deleted neighborhood of the origin.

2. For the system $\dot{x} = 2xy - x^3, \quad \dot{y} = -x^2 - y^5$, show that the function $L(x, y) = x^2 + y^2$ *cannot* be used as a Lyapunov function to show that the origin is an asymptotically stable equilibrium. Try to adjust L so that it can be used for this purpose.

3. Consider the differential equation $\ddot{x} + b\dot{x} - x + x^3 = 0$, $b > 0$. (We have added damping to Problem 1(b) of Homework 2. Your work on that problem will help you do this one.)

   (a) Convert to a system by letting $y = \dot{x}$.

   (b) Use a Lyapunov function suggested by your work on Problem 1(b) of Homework 2, together with Lasalle's Invariance Principle, to prove that every solution approaches one of the equilibria as $t \to \infty$.

4. Consider the differential equation $\ddot{x} + x - x^3 = 0$.

   (a) Convert to a system by letting $y = \dot{x}$.

   (b) Find the equilibria of the system.

   (c) Find a function that is constant on solutions (the Hamiltonian or total energy).

   (d) Draw the phase portrait of the system.

5. Consider the differential equation $\ddot{x} + b\dot{x} + x - x^3 = 0$, $b > 0$. (We have added damping to Problem 4. Your work on that problem will help you do this one.)

   (a) Convert to a system by letting $y = \dot{x}$.

   (b) Define an open set G in R^2 such that if $p \in G$, then the solution through p approaches the origin as $t \to \infty$.

(c) Use a Lyapunov function suggested by your work on Problem 2, together with Lasalle's Invariance Principle, to *prove* that if $p \in G$, then the solution through p approaches the origin as $t \to \infty$.

6. Let $\dot{x} = f(x)$ be a C^1 differential equation on R^n, and let $x \in R^n$. Let $\Gamma = \{\varphi(t, x) : t \geq 0\}$, the positive semi-orbit through x. We use $\mathrm{cl}(A)$ to denote the closure of a set A. Recall that the closure of a set A is the smallest closed set that contains A. It is also A together with all limits of sequences in A.

   (a) Prove: $\mathrm{cl}(\Gamma) = \Gamma \cup \omega(x)$. Suggestion: The harder part is to show that the $\mathrm{cl}(\Gamma)$ is contained in $\Gamma \cup \omega(x)$. Let y be a point in $\mathrm{cl}(\Gamma)$ that is not in Γ. Then there is a sequence of times $t_i \geq 0$ such that $\phi(t_i, x) \to y$. Consider separately the two cases (1) the set of times t_i is bounded and (2) the set of times t_i is unbounded.

   (b) Prove: If $\omega(x)$ is empty, then Γ is closed. (A simple consequence of (a)!)

   (c) Prove: If Γ is compact, then $\Gamma = \omega(x)$.