MA 532 Homework 8

S. Schecter

October 24, 2008

1. Let $x(t)$ be a solution of the differential equation $\dot{x} = Ax + f(t)$, $x \in \mathbb{R}^n$, where (1) all eigenvalues of the $n \times n$ matrix A have negative real part, and (2) the function $f : \mathbb{R} \to \mathbb{R}^n$ is continuous and bounded. Prove that $x(t)$ is bounded on the interval $0 \leq t < \infty$. Suggestion: use Meiss's Lemma 2.9 (p. 58), which says that $\|e^{tA}\| \leq Ke^{-\alpha t}$ for some $K > 0$ and $\alpha > 0$, and the variation of parameters formula (Homework 6, problem 3).

2. Suppose $F : \mathbb{R} \to \mathbb{R}$ is a C^1 positive periodic function with period $p > 0$:

$$F(x + p) = F(x) \text{ for all } x. \tag{1}$$

   Consider the differential equation

$$\frac{dx}{dt} = F(x). \tag{2}$$

   Let $x(t)$ be the solution with $x(0) = 0$. Then $x(t)$ is an increasing function of t (because $\frac{dx}{dt} > 0$) and $x(t)$ is defined for all t (because F, being continuous and periodic, is bounded).

   (a) Explain why $x(t) \to \infty$ as $t \to \infty$.

   (b) Let T be a positive number such that $x(T) = p$. (Such a number exists by (a).) Let $y(t) = x(t + T) - x(T)$. Show that $y(t)$ is a solution of (2).

   (c) Explain why uniqueness of solutions implies that $y(t) = x(t)$ for all t. Conclude that $x(t + T) = x(t) + p$ for all t.

   (d) Explain why

$$\int_0^p \frac{1}{F(x)}\, dx = T.$$

   Suggestion: $x(t)$ is an increasing function of t, so it has an inverse function $t(x)$, and

$$\frac{dt}{dx} = \frac{1}{F(x)}.$$

   Integrate both sides from $x = 0$ to $x = p$.

3. For each integer p, construct the flow $\phi(t, y)$ of $\dot{x} = x^p$. (You will need to treat the case $p = 1$ separately.) Then, for each flow that you construct, verify that $\phi(t + s, y) = \phi(t, \phi(s, y))$.

4. The differential equation $\ddot{u} + \omega^2 u = 0$, rewritten as a system with $x = u$ and $y = \dot{u}$, becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x \end{aligned}$$

Find the flow $\phi(t, (x_0, y_0))$. Then verify that $\phi(t + s, (x_0, y_0)) = \phi(t, \phi(s, (x_0, y_0)))$.