MA 532 Homework 7

S. Schecter

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1. Consider the differential equation $\dot{x} = -\sqrt{|x|}$. Find all solutions that satisfy $x(0) = 0$. (This is similar to the initial value problem $\dot{x} = x^{\frac{2}{3}}$, $x(0) = 0$ that we discussed in class. Suggestion: when $x < 0$, $|x| = -x$.)

2. Let S be a set and let X be a Banach space with norm $\|\cdot\|_X$. Let B denote the set of all bounded functions from S to X. ($f : S \to X$ is in B if there is a number M such that for all $s \in S$, $\|f(s)\|_X \leq M$.) On B define $\|f\| = \sup_{s\in S} \|f(s)\|_X$. Show:

   (a) B, with the appropriate operations, is a vector space. (The only questions are whether the sum of two elements in B is in B, and whether scalar multiples of elements of B are in B.)

   (b) $\|\cdot\|$ is a norm on B.

   (c) B with $\|\cdot\|$ is a Banach space.

3. Let $\phi : [a, b] \to \mathbb{R}$ be a continuous function, let $K : [a, b] \times [a, b] \to \mathbb{R}$ e a continuous function, and let $\lambda > 0$. Define

$$T : C^0([a, b,], \mathbb{R}) \to C^0([a, b,], \mathbb{R})$$

   by

$$T(f)(x) = \lambda \int_a^b K(x, y)f(y)\, dy + \phi(x).$$

   (a) Let C be a number such that $|\phi(x)| \leq C$ for all $x \in [a, b]$. Let M be a number such that $K(x, y) \leq M$ for all $(x, y) \in [a, b] \times [a, b]$. Let $A = \{f \in C^0([a, b,], \mathbb{R}) : \|f\| \leq 2C\}$. Prove: If $\lambda < \frac{1}{2M(b-a)}$, then T maps A into itself and is a contraction.

   (b) Use the Contraction Mapping Theorem to conclude that if $\lambda < \frac{1}{2M(b-a)}$, then there is a continuous function $f : [a, b] \to \mathbb{R}$ such that for all $x \in [a, b]$,

$$f(x) = \lambda \int_a^b K(x, y)f(y)\, dy + \phi(x).$$

4. Let $\dot{x} = f(x)$ be a C^1 differential equation on R^n. Assume that f is "odd": $f(-x) = -f(x)$ for all $x \in R^n$. Let $\phi(t)$ be the solution with $\phi(0) = a$. Use the existence-uniqueness theorem to prove that the unique solution of $\dot{x} = f(x)$ with $\psi(0) = -a$ is $\psi(t) = -\phi(t)$.