MA 532 Homework 6

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October 3, 2008

1. This problem is about the transpose of a matrix, $A^\top$. Recall that $(A+B)^\top = A^\top + B^\top$; $(cA)^\top = cA^\top$; and $(A_1A_2 \dots A_k)^\top = A_k^\top \dots A_2^\top A_1^\top$ (in other words, the transpose of a product is the product, in reverse order, of the transposes).

   Let A be an $n \times n$ matrix. A is *skew-symmetric* if $A^\top = -A$, and A is *orthogonal* if $A^\top A = I$.

   (a) Using the infinite series definition, show that $(e^{tA})^\top = e^{tA^\top}$.

   (b) Using part (a), show: If S is skew-symmetric, then e^{tS} is orthogonal.

   (c) Let S be a 3×3 skew-symmetric matrix whose entries are not all 0.

      i. Show that the eigenvalues of S are 0 and $\pm\beta i$, $\beta > 0$. Hint:
      $$S = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

      ii. Let $x(t)$ be a solution of $\dot{x} = Sx$. Show that $x(t + \frac{2\pi}{\beta}) = x(t)$.

      iii. Let v be an eigenvector of S for the eigenvalue 0. Try to explain intuitively why multiplication by e^{tS} is just rotation about the direction v, through an angle of size βt.

   (d) Let v by a vector in R^3 with $v \neq 0$. Consider the differential equation $\dot{x} = v \times x$. (The $\times$ means cross product.)

      i. Find a 3×3 matrix S such that $\dot{x} = Sx$. Show that S is skew-symmetric.

      ii. By (c), the eigenvalues of S are 0 and $\pm\beta i$. What is the eigenvector for the eigenvalue 0? How is β related to the length of v?

2. Let A be an invertible $n \times n$ matrix, and let b be a vector in $\mathbb{R}^n$. Consider the differential equation
   $$\dot{x} = Ax + b, \quad x(0) = x_0. \tag{1}$$

   (a) Show that the change of coordinates $u = x + A^{-1}b$ converts this differential equation to
   $$\dot{u} = Au, \quad u(0) = x_0 + A^{-1}b. \tag{2}$$

   (b) The solution of (2) is of course $u = e^{tA}u(0)$. By undoing the change of coordinates find the solution of (1).

3. Let A be an $n \times n$ matrix, and let f be a continuous function from an interval J that contains 0 to $\mathbb{R}^n$. Consider the differential equation

$$\dot{x} = Ax + f(t), \quad x(0) = x_0. \tag{3}$$

   (a) Show that the solution of (3) is

$$x = e^{tA}x_0 + \int_0^t e^{(t-s)A} f(s)\, ds. \tag{4}$$

   Suggestion: Rewrite (3) as $\dot{x} - Ax = f(t)$, multiply both sides by e^{-tA}, recognize the left-hand side as a derivative, and integrate.

   (b) In (3), suppose A is invertible and $f(t)$ is a constant vector b. Try to rewrite (4) to equal your solution to 2(b).

4. In the course of proving Floquet's Theorem, we showed that if C is an invertible $n \times n$ matrix, then there is a matrix L, possibly complex, such that $e^L = C$.

   Another version of this result states that if C is a real invertible $n \times n$ matrix, then there is a real matrix L such that $e^L = C^2$. In this problem we will prove some steps in this result.

   (a) Let

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

   with α and β real numbers and $\beta > 0$. Let $r = (\alpha^2 + \beta^2)^{\frac{1}{2}}$, and choose θ such that $\cos\theta = \alpha/r$ and $\sin\theta = \beta/r$. Then $r > 0$ and

$$J = r\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

   Let $K = \ln r I + \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$. Show that $e^K = J$.

   (b) Let

$$C = \begin{pmatrix} J & I \\ 0 & J \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & J^{-1} \\ 0 & 0 \end{pmatrix}\right).$$

   Find a real matrix L such that $e^L = C$. (Part (a) and work we did in class when we proved Floquet's Theorem should help.)