MA 532 Homework 4

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1. Consider the linear differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are continuous on an interval J. Suppose we know one solution

$$\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} \tag{2}$$

with $\phi_1(t_0) \neq 0$ and $\phi_2(t_0) \neq 0$. Let $\alpha(t)$ and $\beta(t)$ be differentiable functions defined on J such that

$$\begin{pmatrix} \alpha(t)\phi_1(t) \\ \beta(t)\phi_2(t) \end{pmatrix} \tag{3}$$

is a second, linearly independent solution on J.

(a) Explain why for all $t \in J$, $\alpha(t) \neq \beta(t)$.

(b) Let

$$W(t) = \det \begin{pmatrix} \alpha(t)\phi_1(t) & \phi_1(t) \\ \beta(t)\phi_2(t) & \phi_2(t) \end{pmatrix} = (\alpha(t) - \beta(t))\phi_1(t)\phi_2(t). \tag{4}$$

Show that

$$\begin{pmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \end{pmatrix} = W(t) \begin{pmatrix} -b(t)/\phi_1(t)^2 \\ c(t)/\phi_2(t)^2 \end{pmatrix}. \tag{5}$$

(c) Show that

$$W(t) = (\alpha(t_0) - \beta(t_0))\phi_1(t_0)\phi_2(t_0)e^{\int_{t_0}^t a(s)+d(s)\,ds}. \tag{6}$$

Hint: Liouville's Formula.

(d) Of course $\alpha(t)$ and $\beta(t)$ are not unique: they can, for example, be multiplied by constants. Thus $W(t_0)$ is arbitrary (but not 0). If you choose, for example, $W(t_0) = 1$, so that

$$W(t) = e^{\int_{t_0}^t a(s)+d(s)\,ds}, \tag{7}$$

and substitute (7) into (5), then the differential equation (5) can be integrated to produce $\alpha(t)$ and $\beta(t)$. There will be two arbitrary constants c_1 and c_2. The fact that $W(t_0) = 1$ will let you solve for one of these constants in terms of the other. The remaining constant can be chosen to make the solution as simple as possible.

Use this approach to find a second, linearly independent solution of

$$\dot{x} = \begin{pmatrix} 1 & -1/t \\ 1+t & -1 \end{pmatrix} x, \quad t > 0,$$

given that one solution is $x(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$.

2. Let A be an invertible $n \times n$ matrix. Show that for the operator norm, $\|A^{-1}\| \geq 1/\|A\|$. Hint: $AA^{-1} = I$.

3. Let A and B be $n \times n$ matrices that commute. This problem gives another proof that $e^{t(A+B)} = e^{tA}e^{tB}$.

   (a) Show that Be^{tA} and $e^{tA}B$ are both solutions of the matrix differential equation $\dot{X} = AX$, with the initial condition $X(0) = B$. (Just substitute and check.) Conclude from the existence-uniqueness theorem for linear differential equations that $Be^{tA} = e^{tA}B$.

   (b) Using part (a), show that $e^{tA}e^{tB}$ is a solution of the matrix differential equation $\dot{X} = (A+B)X$, with the initial condition $X(0) = I$. (Just substitute and check. You will need the product rule for matrix multiplication: $\frac{d}{dt}(X(t)Y(t)) = \dot{X}(t)Y(t) + X(t)\dot{Y}(t)$.) Since another solution is $e^{t(A+B)}$, conclude that $e^{t(A+B)} = e^{tA}e^{tB}$.

4. Show that if $J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, then $e^{tJ} = e^{t\lambda} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$.