MA 532 Homework 10

S. Schecter

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1. Consider the differential equation

$$\dot{x} = x - y^3,$$
$$\dot{y} = 16 - xy.$$

   (a) Find the equilibria.

   (b) Draw the nullclines and the vector field on the nullclines. Using this information, try to draw the phase portrait. Notice that at the equilibrium $(-8, -2)$, solutions appear to spiral around, but you can't tell if the equilibrium is an attractor or a repeller.

   (c) Linearize at the equilibria and identify their types (attractor, repeller, or saddle). Use this information to improve your phase portrait. (You don't need to find the eigenvectors at the saddle, but you you should at least draw its stable and unstable manifolds in a way that is consistent with the vector field in part (b).)

   (d) Use Maple or a similar program to produce a phase portrait. Choose the size of the region so that you can compare to your hand-drawn phase portrait.

2. Consider the Lorenz system

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = xy - bz,$$

with parameters $\sigma > 0$, $r > 0$, and $b > 0$.

   (a) Linearize at the equilibrium $(0, 0, 0)$. For which values of the parameters is the origin a hyperbolic equilibrium? How many eigenvalues have positive real part and how many have negative real part? (To answer the last question, you will have to divide into cases.)

   (b) Use the function $L(x, y, z) = \frac{1}{2}(\frac{x^2}{\sigma} + y^2 + z^2)$ to show that for $0 < r < 1$, the origin is globally asymptotically stable. Hint: First calculate that $\dot{L} = -x^2 + (r + 1)xy - y^2 - bz^2$. Then show that if $0 < r < 1$, then $-x^2 + (r+1)xy - y^2$ is negative for $(x, y) \neq (0, 0)$. One way to do this is to note that for $y = 0$ this is expression is just $-x^2$, and for $y \neq 0$ we can let $z = \frac{x}{y}$ and write $-x^2 + (r + 1)xy - y^2 = -y^2(z^2 - (r + 1)z + 1)$. At this point the quadratic formula should help.

3. Let A be an $n \times n$ matrix all of whose eigenvalues have negative real part. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be bounded and satisfy a Lipschitz condition, *i.e.*, there is a constant $L > 0$ such that

$$\|g(x_1) - g(x_2)\| \leq L\|x_1 - x_2\|$$

for all x_1 and x_2 in $\mathbb{R}^n$. Let x_0 be a point of $\mathbb{R}^n$. We shall use the Contraction Mapping Theorem to show that if L is small enough, then there is a unique solution to the initial value problem

$$\begin{aligned} \dot{x} &= Ax + g(x), \\ x(0) &= x_0, \end{aligned}$$

on the interval $0 \leq t < \infty$, and the solution is bounded.

Our proof begins: If x is a continuous function from $0 \leq t < \infty$ into $\mathbb{R}^n$, define a new continuous function $T(x)$ from $0 \leq t < \infty$ into $\mathbb{R}^n$ by the formula

$$T(x)(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} g(x(s))\, ds.$$

(a) Use Homework 8, Problem 1 to show that $T(x)$ is a bounded function of t.

(b) Show that if L is small enough, then $T : C^0([0, \infty), \mathbb{R}^n) \to C^0([0, \infty), \mathbb{R}^n)$ is a contraction. How small must L be?

(c) Explain why the the fixed point of the contraction is a bounded solution of the initial value problem.