MA 440 Final Exam

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1. Ajax Industries and Babar Industries both produce widgets. Let x_1 be the quantity produced by Ajax and let x_2 be the quantity produced by Babar. The price of widgets, in dollars, is

$$p = 100 - 2(x_1 + x_2).$$

Thus the more widgets the two companies produce, the lower will be the price. (This formula can produce a negative price; don't worry about it.)

It costs Ajax industries \$3 to make each widget, and it costs Babar Industries \$2 to make each widget.

The revenue of each company is the price times the quantity it produces. Profit is revenue minus cost of production. Thus the profit of Ajax Industries is

$$\pi_1 = px_1 - 3x_1 = 97x_1 - 2x_1^2 - 2x_1x_2,$$

and the profit of Babar Industries is

$$\pi_2 = px_2 - 2x_2 = 98x_2 - 2x_1x_2 - 2x_2^2.$$

We regard this as a two-person game. The players choose x_1 and x_2; the payoffs are π_1 and π_2.

(a) Suppose the two players simultaneously choose the quantities they produce. Find the Nash equilibrium.

(b) Suppose Ajax chooses x_1 first, then Babar observes x_1 and chooses x_2. Use backward induction to find Ajax's best choice.

If your answer involves fractions of a widget, don't worry about it.

2. Sub Station has the only sub restaurant in Town A and the only sub restaurant in Town B. The sub market in each town yields a profit of \$100K per year. Rival Sub Machine is considering opening a restaurant in Town A in year 1. If it does, the two stores will split the profit from the sub market there. However, Sub Machine will have to pay setup costs for its new store. These costs are \$25K in a store's first year.

   Sub Station fears that if Sub Machine is able to make a profit in Town A, it will open a store in Town B the following year. Sub Station is considering a price war: if Sub Machine opens a store in either town, it will lower prices in that town, forcing Sub Machine to do the same, to the point where profits from the sub market in that town drop to 0.

   The following game tree is one way to represent the situation. It takes into account net profits from Towns A and B in years 1 and 2, and it assumes that if Sub Machine loses money in A it will not open a store in B.

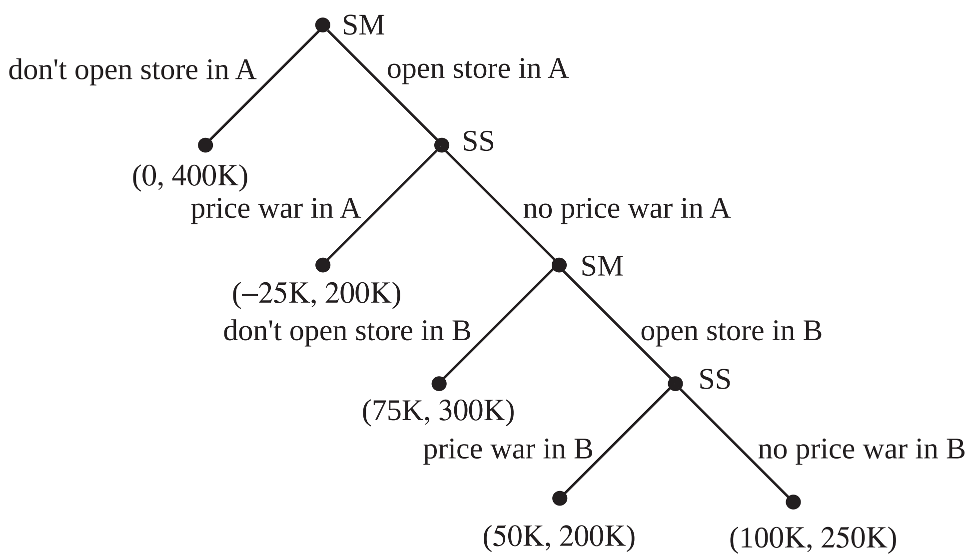


Figure 1: SM is Sub Machine, SS is Sub Station. Sub Machine's profits are first, Sub Station's profits are second.

   For example, the entry $(100\text{K}, 250\text{K})$ in the table comes about as follows. If there are no price wars, Sub Machine makes net profits of \$25K in Town A in year 1, \$50K in Town A in year 2, and \$25K in Town B in year 2, for a total of \$100K. Sub Station makes profits of \$50K in Town A in year 1, \$50K in Town A in year 2, \$100K in Town B in year 1, and \$50K in Town B in year 2, for a total of \$250K.

(a) Explain the entry $(50\mathrm{K}, 200\mathrm{K})$ in the table.

(b) Use backward induction to figure out what Sub Machine and Sub Station should do. Be sure I can follow your reasoning.

3. Two countries each have one unit of wealth. Each chooses a fraction of its wealth to devote to fighting the other. The country that devotes a larger fraction of its wealth to fighting wins the fight. Its payoff is the remaining wealth of both countries. The losing country's payoff is zero. If both counties devote the same fraction of their wealth to fighting, the result is a tie. In this case, each country's payoff is its remaining wealth.

   We consider this situation as a two-person game. The first country's strategy is a number s, $0 \leq s \leq 1$, that represents the fraction of its wealth it will devote to fighting. Similarly, the second country's strategy is a number t, $0 \leq t \leq 1$, that represents the fraction of its wealth it will devote to fighting. We assume the two countries choose their strategies simultaneously.

   The payoffs are:

   - If $s < t$, $\Pi_1(s,t) = 0$ and $\Pi_2(s,t) = 2 - (s+t)$.
   - If $s > t$, $\Pi_1(s,t) = 2 - (s+t)$ and $\Pi_2(s,t) = 0$.
   - If $s = t$, $\Pi_1(s,t) = 1-s$ and $\Pi_2(s,t) = 1-t$. Of course, $1-s = 1-t$.

   (a) Find all Nash equilbria with $s < t$. You may need to consider separately the case $t = 1$.

   (b) Find all Nash equilibria with $s = t$.

   On both problems, make sure I can follow your reasoning. For each strategy profile (s,t), you need to explain why it is or is not a Nash equilibrium.

4. In the nation of Slobovia, two companies make breakfast cereal. Breakfast cereal can be made in five sweetness levels, numbered 1 to 5. Each sweetness level is preferred by 20% of the population. If a person must choose between two available sweetness levels, she chooses the one closest to her preference, and chooses randomly if the available sweetness level are equally close to her preference.

   We model this situation as a two-player game. The players are the two companies; each company's strategy is the sweetness of the cereal it produces; each company's payoff is the percentage of the population that chooses its cereal. We assume the companies choose simultaneously. The following matrix gives the payoffs.

		Company 2				
		1	**2**	**3**	**4**	**5**
	1	(50,50)	(20,80)	(30,70)	(40,60)	(50,50)
	2	(80,20)	(50,50)	(40,60)	(50,50)	(60,40)
Company1	**3**	(70,30)	(60,40)	(50,50)	(60,40)	(70,30)
	4	(60,40)	(50,50)	(40,60)	(50,50)	(80,20)
	5	(50,50)	(40,60)	(30,70)	(20,80)	(50,50)

   (a) Explain the five entries in the second column.

   (b) Use iterated elimination of strictly dominated strategies to find a Nash equilibrium. Be sure to make clear the order in which you eliminate strategies, and which strategy dominates each strategy that you eliminate.

5. Two drivers arrive at an intersection coming from opposite directions. Each wants to turn left. Each has three strategies:

   - T: turn.
   - W: wait.
   - C: contingent: turn if the other motorist seems to be waiting, wait if the other motorist seems to be turning.

   We assume the payoffs are as follows:

		Driver 2		
		T	**W**	**C**
	T	$(-5,-5)$	$(0,-2)$	$(0,-2)$
Driver 1	**W**	$(-2,0)$	$(-3,-3)$	$(-2,0)$
	C	$(-2,0)$	$(0,-2)$	$(-5,-5)$

(a) Use best response to find all pure strategy Nash equilibria.

(b) Find a mixed strategy Nash equilibrium in which both drivers use all three strategies with positive probability. (Because of the symmetry of the problem, once you have found the three probabilities for one driver, you may assume that the other driver uses the same three probabilities.)

(c) Try to find a Nash equilibrium in which both drivers use strategies T and C with positive probability, and strategy W with 0 probability. (Again, because of the symmetry of this problem, once you have found the two probabilities for one driver, you may assume that the other driver uses the same two probabilities.) Don't forget the final step in checking that you really have a Nash equilibrium.

6. In a certain town there are two sub shops, Sub Station and Sub Machine. Each can charge high prices or low prices. If both charge high prices, both make \$4K per week. If both charge low prices, both make \$3K per week. If one charges high prices and one charges low prices, the one charging high prices makes nothing; the one charging low prices makes \$6K per week.

   Suppose both shops use the following trigger strategy:

   - Start by charging high prices. If the other shop charges high prices in period k, then charge high prices in period $k + 1$. If the other shop charges low prices in period k, then charge low prices in period $k + 1$ and in every subsequent period.

   The discount factor is δ, $0 < \delta < 1$.

   Find a number δ_0 such that if $\delta \geq \delta_0$, it is a Nash equilibrium for both shops to use this trigger strategy.

7. I often face the following problem. Something of mine does not work properly (car, computer, body). I take it to an Expert (mechanic, computer repair person, doctor). The problem may be major or minor. The Expert studies the problem and diagnoses it as major or minor. I then must decide whether to follow the Expert's advice and do the repair.

   - Expert's payoffs
     - Bill for major repair: M.
     - Bill for minor repair: m.
     - Boost to reputation from making correct diagnosis: B.
   - Customer's payoffs
     - Value of getting major problem fixed: V.
     - Value of getting minor problem fixed: v.
     - Bill for major repair: $-M$.
     - Bill for minor repair: $-m$.

   Half the time the problem is minor, and half the time it is major; everyone knows this.

   If the problem is major, a minor repair will not fix it. If the problem is minor, either a minor or a major repair will fix it.

   We assume $V > M > v > m$ and $B > m$.

   The following game tree illustrates the situation.

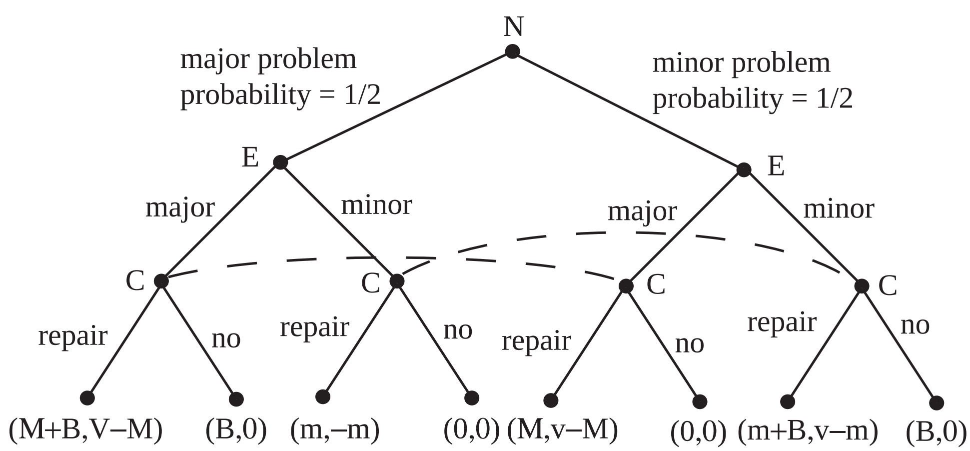


Figure 2: N = Nature, E = Expert, C = Customer. The Expert diagnoses a major or a minor problem; then the Customer decides whether to do the repair or not. The first payoff is to the Expert, the second is to the Customer.

The Customer has four strategies:

- Rr: If Expert says problem is major, repair; if Expert says problem is minor, repair.
- Rn: If Expert says problem is major, repair; if Expert says problem is minor, do not repair.
- Nr: If Expert says problem is major, do not repair; if Expert says problem is minor, repair.
- Nn: If Expert says problem is major, do not repair; if Expert says problem is minor, do not repair.

(a) Explain why the Expert should always diagnose a major problem as major.

(b) Because of part (a), we assume the Expert always diagnoses a major problem as major. Therefore the Expert only has two pure strategies:

- D: Dishonest: diagnose every problem as major.
- H: Honest: if the problem is major, diagnose it as major; if the problem is minor, diagnose it as minor.

Complete the following 2×4 payoff matrix, showing expected payoffs to both Expert and Customer.

		Customer			
		Rr	**Rn**	**Nr**	**Nn**
Expert	**Dishonest**				
	Honest				

8. When two little monkeys encounter a warifruit tree, one of them may climb up and knock down a fruit. Then both monkeys eat it. The monkey that climbs the tree gets less to eat (because the other monkey has a head start) and incurs an energy cost.

   The payoffs are as follows:

		Monkey 2	
		climb	**wait**
Monkey 1	**climb**	$(2,2)$	$(1,4)$
	wait	$(4,1)$	$(0,0)$

   (a) Find the pure strategy Nash equilibria. Are there any that are symmetric?

   (b) Find all mixed strategy Nash equilibria. (There is one and it is symmetric.)

   (c) Check whether your mixed strategy Nash equilibrium gives an evolutionarily stable state.

   (d) Derive the replicator system and reduce it to a single differential equation.

   (e) Find all equilibria of your differential equation.

   (f) Draw the phase portrait of your differential equation.

9. When two big monkeys encounter a warifruit tree, the situation is different. For one thing, if both monkeys try to climb the tree, they will get in each other's way and will fall down. On the other hand, two big monkeys are strong enough to shake the tree together and knock down a fruit. However, if one monkey climbs and one shakes, the shaking prevents the first monkey from successfully climbing, but is not sufficient to knock down a fruit.

   The payoffs are as follows:

		Monkey 2		
		climb	**wait**	**shake**
	climb	$(0,0)$	$(1,3)$	$(0,0)$
Monkey 1	**wait**	$(3,1)$	$(0,0)$	$(0,0)$
	shake	$(0,0)$	$(0,0)$	$(2,2)$

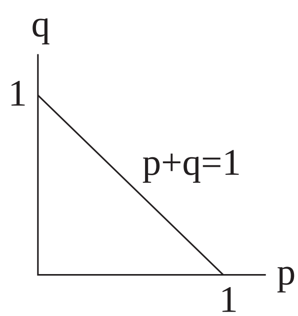
   (a) Find the pure strategy Nash equilibria. Are there any that are symmetric?

(b) Denote a population type by $\sigma = pc + qw + rs$ with $p \geq 0$, $q \geq 0$, $r \geq 0$, and $p + q + r = 1$. Show that the replicator equation, using the variables p and q only, is

$$\begin{aligned} \dot{p} &= (q - h(p,q))p, \\ \dot{q} &= (3p - h(p,q))q, \end{aligned}$$

with $h(p,q) = 4pq + 2(1-p-q)^2$.

(c) We study this differential equation on the following triangle.



Find three equilibria in the triangle that lie on the line $p + q = 1$.

(d) Compute the eigenvalues of the linearization at the equilibrium $(0,0)$ and describe its type (attractor, repeller, saddle).

(e) The phase portrait is shown below. What does it tell you?