MA 426/591M Test 2

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1. Let
$$f(x,y) = \begin{cases} \frac{x^2y}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Using the definition of derivative, prove: $Df(0,0) = [0\ \ 0]$. Hint: $x^2 \leq x^2 + y^2$.

2. Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ is
$$f(x_1,x_2,x_3) = (x_1^2x_2x_3, x_2^2 + x_3^2),$$
and $g : \mathbb{R} \to \mathbb{R}^3$ is
$$g(t) = (2t+1, e^t, 4).$$
   (a) Calculate $Df(x_1,x_2,x_3)$ and $Dg(t)$.
   (b) Using the chain rule that we learned this semester, which involves multiplication of matrices, calculate
   $D(f \circ g)(0)$.

3. Consider the system of equations
$$(x^2 + u^2)(y^2 + v^2) = 1,$$
$$x \cos u + y \sin v = 1.$$
   (a) Show that the Implicit Function Theorem implies we can solve for (x,y) in terms of (u,v) near $(x,y,u,v) = (1,1,0,0)$.
   (b) Compute the matrix of partial derivatives of (x,y) with respect to (u,v) at that point.

Do *two* of the following three problems.

(4) Prove: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x_0, then f is continuous at x_0.

(5) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, $Df(0) = 0$, and the bilinear form associated with $D^2 f(0)$ is negative definite. Prove that f has a local maximum at $x = 0$. (In your proof you may use Taylor's formula and the lemma that says: if B is a negative definite bilinear form, then there is a constant $m > 0$ such that $B(x, x) \leq -m\|x\|^2$ for all $x \in \mathbb{R}^n$).

(6) Let $A \subset \mathbb{R}^n$ be convex. (This means that if x and y are in A, then the entire line segment that joins them is in A.) Let $f : A \to \mathbb{R}^m$ be C^1. Suppose there is a number $M > 0$ such that for all $x \in A$ and all $z \in \mathbb{R}^n$, $\|Df(x)z\| \leq M\|z\|\|$. Prove: If $x \in A$ and $y \in A$ then $\|f(x)-f(y)\| \leq M\|y-x\|$. Suggestions: (1) Let $h(t) = f(x+t(y-x))$. (2) You may use the following: If $g : [a, b] \to \mathbb{R}^m$ is continuous, then $\| \int_a^b g(t)dt\| \leq \int_a^b \|g(t)\|dt$.