MA 426/591M Homework

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Assigned March 21, 2003, due March 28, 2003

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Show f is continuous at $(0,0)$. Suggestion: $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$.

   (b) Find all directional derivatives of f at $(0,0)$ that exist.

   (c) Is f differentiable at $(0,0)$? Justify your answer.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{x^2y}{x^6+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Show that all directional derivatives of f at $(0,0)$ exist.

   (b) Show that f not continuous at $(0,0)$. Suggestion: Look at the values of f on the curve $\phi(t) = (t, t^3)$, which goes through the origin.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Show that f is differentiable at $(0,0)$ and $Df(0,0) = 0$.

   (b) Calculate $\frac{\partial f}{\partial x}(0,y)$. In other words, at each point on the y-axis, find the partial derivative of f with respect to x. (Unless $y = 0$, you can just do this using the quotient rule.)

(c) Calculate $\frac{\partial f}{\partial y}(x, 0)$. In other words, at each point on the x-axis, find the partial derivative of f with respect to y. (Unless $x = 0$, you can just do this using the quotient rule.)

(d) Using your answers to parts (b) and (c) show that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

(e) For this function, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist at every point (x, y). Do you think that they are continuous at $(0,0)$? Hint: Equality of mixed partials.

4. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *homogeneous of degree* n if for all (x, y) in $\mathbb{R}^2$ and for all $t > 0$,

$$f(tx, ty) = t^n f(x, y).$$

(a) Show that the functions $f(x, y) = \frac{xy}{x+y}$, $f(x, y) = \frac{x}{x^2+y^2}$, and $f(x, y) = x^{\frac{1}{3}} + x^{-\frac{2}{3}} y$ are all homogeneous. What are the degrees?

(b) Prove *Euler's Theorem*: If $f : \mathbb{R}^2 \to \mathbb{R}$ is homogeneous of degree n, then at any point (x, y) where f is differentiable,

$$x f_x(x, y) + y f_y(x, y) = n f(x, y).$$

Suggestion. *Fix* a point (x, y) where f is differentiable, and define $\phi : \mathbb{R} \to \mathbb{R}^2$ by $\phi(t) = (tx, ty)$. (Remember, (x, y) is fixed.) Consider the composition $f \circ \phi(t) = f(tx, ty)$. Calculate the derivative of $f \circ \phi$ at $t = 1$ two different ways, one with the chain rule, one without the chain rule.