MA 425-002 Practice Problems for Test 2

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Give proofs from first principles.

1. Suppose (x_n) is a Cauchy sequence. Without using the fact that a Cauchy sequence converges, prove that (x_n) is bounded.

2. Let (x_n) and (y_n) be sequences. Assume

   (a) $\lim x_n = \infty$.

   (b) There is a number $M > 0$ such that $0 < y_n < M$ for every n.

   Prove that
   $$\lim \frac{x_n}{y_n} = \infty.$$

3. Define $f : \mathbb{R} \to \mathbb{R}$ by
   $$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 2x & \text{if } x \text{ is irrational.} \end{cases}$$
   Prove that $\lim_{x \to 0} f(x) = 0$.

4. Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ be functions. Suppose $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = L$, where L is a finite number. Prove that $\lim_{x \to a} f(x) + g(x) = \infty$.

5. Let $f : A \to \mathbb{R}$ be a function and let $c \in A$. Assume that f is continuous at $x = c$. Suppose that $f(c) = d$ and $d > 0$. Prove that there is a number $\delta > 0$ such that if $x \in A$ and $|x - c| < \delta$ then $f(x) > \frac{2}{3}d$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that $g \circ f$ is continuous.

7. Let f be a continuous function on the interval $(0,1)$ such that $f(x) > 0$ for all x in $(0,1)$. Is it possible that

$$\inf_{x \in (0,1)} f(x) = 0?$$

   (a) Yes.

   (b) No.

8. Let f be a continuous function on the interval $[0,1]$ such that $f(x) > 0$ for all x in $[0,1]$. Is it possible that

$$\inf_{x \in (0,1)} f(x) = 0?$$

   (a) Yes.

   (b) No.

9. Let $f : (1,\infty) \to \mathbb{R}$ be $f(x) = \frac{1+x}{2x}$. Prove that f is uniformly continuous.

10. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be uniformly continuous functions. Prove that $f + g$ is uniformly continuous.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for every x, $-x^2 \le f(x) \le x^2$. Prove that f is differentiable at 0, and $f'(0) = 0$. (Notice that $f(0)$ has to be 0. Be careful with this problem. If you divide by x when x is negative, inequalities reverse.)

12. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Assume that f' is a strictly increasing function on $[a,b]$. (This means: If $x_1 \in [a,b]$, $x_2 \in [a,b]$, and $x_1 < x_2$, then $f'(x_1) < f'(x_2)$.) Prove: $f(b) - f(a) - f'(a)(b-a) > 0$. Hint: What does the Mean Value Theorem tell you about $f(b) - f(a)$?