MA 425-002 Final Exam

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Do eight problems. The answers to problems 1, 3, 4, 5 and 8 should include the expression, "Let $\epsilon > 0$."

1. Let $x_n = \frac{2n^2}{1+n^2}$. Prove that $x_n \to 2$.

2. Let (x_n) be a sequence such that $x_n \to x$. Suppose $x < 0$. Prove that there is a number N such that $x_n < 0$ for all $n > N$.

3. Prove: If (x_n) is a bounded decreasing sequence and $u = \inf\{x_n : n \in \mathbb{N}\}$, then $x_n \to u$.

4. Let $f : (0, \infty) \to \mathbb{R}$ and $g : (0, \infty) \to \mathbb{R}$ be functions. Assume:

   (a) $f(x) > 0$ for all x.

   (b) $\lim_{x \to 0} f(x) = \infty$.

   (c) g is a bounded function.

   Prove that $\lim_{x \to 0} \frac{g(x)}{f(x)} = 0$.

5. Show that the function $f(x) = \frac{1+x}{x^2}$ is uniformly continuous on the interval $1 \le x < \infty$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for every x, $-x^2 \le f(x) \le x^2$. Prove that f is differentiable at 0, and $f'(0) = 0$. (Notice that $f(0)$ has to be 0. Be careful with this problem. If you divide by x when x is negative, inequalities reverse.)

7. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Assume that f' is a strictly increasing function on $[a, b]$. (This means: If $x_1 \in [a, b]$, $x_2 \in [a, b]$, and $x_1 < x_2$, then $f'(x_1) < f'(x_2)$.) Prove: $f(b) - f(a) - f'(a)(b - a) > 0$. Hint: What does the Mean Value Theorem tell you about $f(b) - f(a)$?

8. Let

$$f_n(x) = \frac{1 + nx^2}{nx}, \quad 0 < x < \infty,$$
$$f(x) = x, \quad 0 < x < \infty.$$

   Show that if $a > 0$, then $f_n \to f$ uniformly on the interval $a \leq x < \infty$.

9. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $c \in (a, b)$. Assume:

   - $f(x) \geq 0$ for all $x \in [a, b]$.
   - $f(c) > 0$.

   Show that $\int_a^b f > 0$.

10. Let (a_n) and (b_n) be positive sequences. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.