

$$\int (f(u) + g(u)) du = \int f(u) du + \int g(u) du. \quad \int cf(u) du = c \int f(u) du.$$

$$\int u dv = uv - \int v du. \quad \int u^n du = \frac{u^{n+1}}{n+1}, \quad n \neq -1. \quad \int \frac{du}{u} = \ln|u|.$$

$$\int e^u du = e^u. \quad \int ue^u du = (u-1)e^u. \quad \int u^n e^u du = u^n e^u - n \int u^{n-1} e^u du.$$

$$\int a^u du = \frac{a^u}{\ln a}, \quad a > 0, a \neq 1. \quad \int \ln u du = u \ln u - u. \quad \int \frac{du}{u \ln u} = \ln|\ln u|.$$

$$\int u^n \ln u du = u^{n+1} \left(\frac{\ln u}{n+1} - \frac{1}{(n+1)^2} \right), \quad n \neq -1. \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a}.$$

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right|. \quad \int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right|.$$

$$\int \frac{du}{(a+bu)(\alpha+\beta u)} = \frac{1}{a\beta - \alpha b} \ln \left| \frac{\alpha + \beta u}{a + bu} \right|$$

$$\int \frac{u du}{(a+bu)(\alpha+\beta u)} = \frac{1}{a\beta - \alpha b} \left[\frac{a}{b} \ln|a+bu| - \frac{\alpha}{\beta} \ln|\alpha+\beta u| \right].$$

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln|u + \sqrt{u^2 + a^2}|. \quad \int \frac{du}{\sqrt{u^2 - a^2}} = \ln|u + \sqrt{u^2 - a^2}|, \quad u^2 \geq a^2.$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a}, \quad a^2 \geq u^2. \quad \int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \arccos \frac{a}{u}, \quad u > a > 0.$$

$$\int \sin u du = -\cos u. \quad \int \cos u du = \sin u. \quad \int \tan u du = -\ln|\cos u|.$$

$$\int \cot u du = \ln|\sin u|. \quad \int \sec u du = \ln|\sec u + \tan u|.$$

$$\int \csc u du = -\ln|\csc u + \cot u| = \ln|\csc u - \cot u|.$$

$$\int \sec^2 u du = \tan u. \quad \int \csc^2 u du = -\cot u. \quad \int \sec u \tan u du = \sec u.$$

$$\int \sin^2 u du = \frac{1}{2}u - \frac{1}{4}\sin 2u. \quad \int \cos^2 u du = \frac{1}{2}u + \frac{1}{4}\sin 2u. \quad \int \tan^2 u du = \tan u - u.$$

$$\int \sin^n u du = -\frac{\sin^{n-1} u \cos u}{n} + \frac{n-1}{n} \int \sin^{n-2} u du.$$

$$\int \cos^n u du = \frac{\cos^{n-1} u \sin u}{n} + \frac{n-1}{n} \int \cos^{n-2} u du.$$

*Note: An arbitrary constant is to be added to each indefinite integral.

$$\int u \sin u du = \sin u - u \cos u. \quad \int u^n \sin u du = -u^n \cos u + n \int u^{n-1} \cos u du.$$

$$\int u \cos u du = \cos u + u \sin u. \quad \int u^n \cos u du = u^n \sin u - n \int u^{n-1} \sin u du.$$

$$\int e^{au} \sin nu du = \frac{e^{au}(a \sin nu - n \cos nu)}{a^2 + n^2}. \quad \int e^{au} \cos nu du = \frac{e^{au}(a \cos nu + n \sin nu)}{a^2 + n^2}.$$

$$\int \sin au \sin bu du = -\frac{\sin(a+b)u}{2(a+b)} + \frac{\sin(a-b)u}{2(a-b)}, \quad a^2 \neq b^2.$$

$$\int \cos au \cos bu du = \frac{\sin(a+b)u}{2(a+b)} + \frac{\sin(a-b)u}{2(a-b)}, \quad a^2 \neq b^2.$$

$$\int \sin au \cos bu du = -\frac{\cos(a+b)u}{2(a+b)} - \frac{\cos(a-b)u}{2(a-b)}, \quad a^2 \neq b^2.$$

$$\int \sinh u du = \cosh u. \quad \int \cosh u du = \sinh u.$$

$$\Gamma(t) = \int_0^\infty e^{-u} u^{t-1} du, \quad t > 0; \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}; \quad \text{and} \quad \Gamma(n+1) = n!, \text{ if } n \text{ is a positive integer.}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \quad (\text{Taylor series})$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n \quad (1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n \quad \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \cdots$$

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(k!)^2 2^{2k}} \quad J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(k+1)! 2^{2k+1}} \quad J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{k! \Gamma(n+k+1) 2^{2k+n}}$$

*Note: An arbitrary constant is to be added to each indefinite integral.

LINEAR FIRST-ORDER EQUATIONS

A general solution to the first-order linear equation $dy/dx + P(x)y = Q(x)$ is

$$y(x) = [\mu(x)]^{-1} \left(\int \mu(x)Q(x)dx + C \right), \quad \text{where } \mu(x) = \exp \left(\int P(x)dx \right).$$

METHOD OF UNDETERMINED COEFFICIENTS

To find a particular solution to the constant-coefficient differential equation

$$ay'' + by' + cy = P_m(t)e^{rt},$$

where $P_m(t)$ is a polynomial of degree m , use the form

$$y_p(t) = t^s(A_mt^m + \dots + A_1t + A_0)e^{rt};$$

if r is not a root of the associated auxiliary equation, take $s = 0$; if r is a simple root of the associated auxiliary equation, take $s = 1$; and if r is a double root of the associated auxiliary equation, take $s = 2$.

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{\alpha t} \cos \beta t + Q_n(t)e^{\alpha t} \sin \beta t,$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n , use the form

$$y_p(t) = t^s(A_k t^k + \dots + A_1 t + A_0) e^{\alpha t} \cos \beta t + t^s(B_k t^k + \dots + B_1 t + B_0) e^{\alpha t} \sin \beta t,$$

where k is the larger of m and n . If $\alpha + i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if $\alpha + i\beta$ is a root of the associated auxiliary equation, take $s = 1$.

VARIATION OF PARAMETERS FORMULA

If y_1 and y_2 are two linearly independent solutions to $ay'' + by' + cy = 0$, then a particular solution to $ay'' + by' + cy = g$ is $y = v_1 y_1 + v_2 y_2$, where

$$v_1(t) = \int \frac{-g(t)y_2(t)}{aW[y_1, y_2](t)} dt, \quad v_2(t) = \int \frac{g(t)y_1(t)}{aW[y_1, y_2](t)} dt,$$

and $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$.

A TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1. $f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$	20. $\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
2. $e^{at}f(t)$	$F(s - a)$	21. \sqrt{t}	$\frac{\sqrt{\pi}}{2s^{3/2}}$
3. $f'(t)$	$sF(s) - f(0)$	22. $t^{n-(1/2)}, \quad n = 1, 2, \dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+(1/2)}}$
4. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$	23. $t^r, \quad r > -1$	$\frac{\Gamma(r+1)}{s^{r+1}}$
5. $t^n f(t)$	$(-1)^n F^{(n)}(s)$	24. $\sin bt$	$\frac{b}{s^2 + b^2}$
6. $\frac{1}{t} f(t)$	$\int_s^\infty F(u) du$	25. $\cos bt$	$\frac{s}{s^2 + b^2}$
7. $\int_0^t f(v) dv$	$\frac{F(s)}{s}$	26. $e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$
8. $(f * g)(t)$	$F(s)G(s)$	27. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$
9. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t) dt}{1 - e^{-sT}}$	28. $\sinh bt$	$\frac{b}{s^2 - b^2}$
10. $f(t-a)u(t-a), \quad a \geq 0$	$e^{-as}F(s)$	29. $\cosh bt$	$\frac{s}{s^2 - b^2}$
11. $g(t)u(t-a), \quad a \geq 0$	$e^{-as}\mathcal{L}\{g(t+a)\}(s)$	30. $\sin bt - bt \cos bt$	$\frac{2b^3}{(s^2 + b^2)^2}$
12. $u(t-a), \quad a \geq 0$	$\frac{e^{-as}}{s}$	31. $t \sin bt$	$\frac{2bs}{(s^2 + b^2)^2}$
13. $\Pi_{a,b}(t), \quad 0 < a < b$	$\frac{e^{-sa} - e^{-sb}}{s}$	32. $\sin bt + bt \cos bt$	$\frac{2bs^2}{(s^2 + b^2)^2}$
14. $\delta(t-a), \quad a \geq 0$	e^{-as}	33. $t \cos bt$	$\frac{s^2 - b^2}{(s^2 + b^2)^2}$
15. e^{at}	$\frac{1}{s-a}$	34. $\sin bt \cosh bt - \cos bt \sinh bt$	$\frac{4b^3}{s^4 + 4b^4}$
16. $t^n, \quad n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}$	35. $\sin bt \sinh bt$	$\frac{2b^2s}{s^4 + 4b^4}$
17. $e^{at}t^n, \quad n = 1, 2, \dots$	$\frac{n!}{(s-a)^{n+1}}$	36. $\sinh bt - \sin bt$	$\frac{2b^3}{s^4 - b^4}$
18. $e^{at} - e^{bt}$	$\frac{(a-b)}{(s-a)(s-b)}$	37. $\cosh bt - \cos bt$	$\frac{2b^2s}{s^4 - b^4}$
19. $ae^{at} - be^{bt}$	$\frac{(a-b)s}{(s-a)(s-b)}$	38. $J_v(bt), v > -1$	$\frac{(\sqrt{s^2 + b^2} - s)^v}{b^v \sqrt{s^2 + b^2}}$